Mathematical Methods and Electrodynamics

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• Vector Algebra

Contents

1. Introduction

The underlying elements in vector analysis are vectors and scalars.

A. Vectors (As directed line segments):

There are quantities in physics and science characterized by both magnitude and direction, such as displacement, velocity, force, and acceleration. Pictorially, these are thus denoted as directed (with arrowheads) straight line segments. The direction of the arrow is the direction of the said quantity and the magnitude of the quantity is proportional to the length of the line segment. In the figure on the left, the line-segment AB denotes a force of $5N$, and has a length $1cm$. The force is from A to B. As the line below that is double in length (i.e. $2\ cm$), it must denote a force of $10N$ in the same direction.

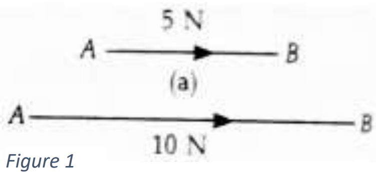

Figure 1

For completeness, a ***Scalar*** is a quantity that is completely denoted by just its magnitude. Examples are mass, length, temperature, etc.

To be considered as a vector, it is not enough for a quantity to have just amplitude and direction. It must also follow certain rules in addition. These rules are the rules of *vector algebra*. Hence, for a physical quantity to be a vector,

1. **The quantity must have a magnitude and a direction, independent of the *frame of reference*.**
2. **It must follow the rules of vector algebra.**

Remember that we will always denote a vector with an arrow on top of it.

Before writing down the rules of vector algebra, let us talk about some properties and definitions regarding vectors.

B. Useful Information about vectors:

1. **Equal Vectors:** *Two vectors are equal only if they have equal magnitude and direction regardless of their initial point.*
   In the first figure on the left, $|\vec{A}| = PQ = |\vec{B}| = MN$ (i.e. their magnitudes are the same) and they have the same direction. Hence, they are equal. On the other hand, though the lengths of the vector-pairs in the other two figures are the same, they have different directions, and hence, are *not equal*.

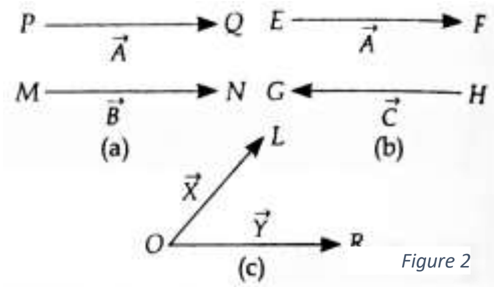

Figure 2

2. **Opposite Vectors:** *Two vectors with the same magnitude but opposite directions are called opposite vectors of each other*.
   Vectors $\vec{A}$ and $\vec{C}$ in the figure (b) above are equal in magnitude, i.e. $|\vec{A}| = |\vec{C}| = EF = GH$, but opposite in direction. $\therefore \vec{A} = -\vec{C}$
3. **Null or Zero Vector:** *When the two end-points of a vector coincide, i.e. its length/magnitude becomes zero and it does not have a specific direction, it is called a null vector*.
   It is represented as $\vec{0}$. Properties of null vector are:

i. $\vec{A} \pm \vec{0} = \vec{A}$
ii. $\vec{A} + (-\vec{A}) = \vec{0}$
iii. $k\,\vec{0} = \vec{0}$, where k is a scalar
iv. $0\,\vec{A} = \vec{0}$

4. **Collinear Vectors:** *Two vectors, with either the same or opposite direction, are called collinear if they lie on the same line or parallel lines*.

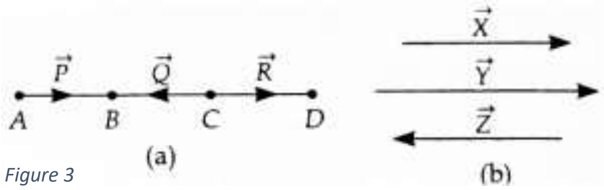

Figure 3

In the attached figure, vectors $\vec{P}, \vec{Q}$, and $\vec{R}$ are on the same straight line, whereas $\vec{X}, \vec{Y}$, and $\vec{Z}$ are parallel to each other. $\vec{P}, \vec{Q}$, and $\vec{R}$ are collinear and $\vec{X}, \vec{Y}$, and $\vec{Z}$ are collinear as well.
Two collinear vectors with the same direction are called ***like vectors*** (Obviously, two like vectors with equal magnitude are equal vectors). In the figure, $\vec{P}$ and $\vec{R}$ are like vectors and same is true for $\vec{X}$ and $\vec{Y}$.

5. **Coplanar Vectors** *are vectors that lie in the same plane in three-dimensional space*.
6. **Unit Vectors** *are vectors with unit length/magnitude, also known as directional vectors*. (Unit vectors are denoted with a 'hat'/'circumflex' i.e. ^ sign on top of them, instead of an arrow.) If a vector $\vec{A}$ has magnitude A and $\hat{a}$ is the unit vector in the direction of $\vec{A}$, then
$\hat{a} = \frac{\vec{A}}{A} \qquad \therefore\ \vec{A} = A\,\hat{a}$

2. Laws of Vector Algebra:

A. Scalar Multiplication

In common geometrical contexts, scalar multiplication of a real Euclidean vector by a positive real number multiplies the magnitude of the vector - without changing its direction. The term 'scalar' itself derives from this usage: a scalar is that which scales vectors. Scalar multiplication is the multiplication of a vector by a scalar (where the product is a vector). If the scalar is negative, then the vector scales in magnitude as well as changes the direction to the opposite side.

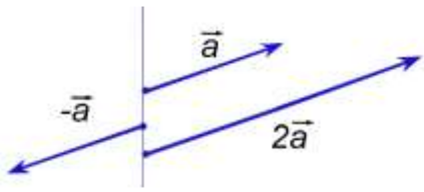

Figure 4
By Silly rabbit - enwiki, CC BY-SA 3.0, https://commons.wikimedia.org/w/index.php?curid=5088002

B. Laws

1. $\vec{A} + \vec{B} = \vec{B} + \vec{A}$ (The **Commutative Law** for Addition)
2. $\vec{A} + (\vec{B} + \vec{C}) = (\vec{A} + \vec{B}) + \vec{C}$ (The **Associative Law** for Addition)
3. When m and n are scalars,
   i. $m(\vec{A} \pm \vec{B}) = m\vec{A} \pm m\vec{B}$
   ii. $(m \pm n)\vec{A} = m\vec{A} \pm n\vec{A}$
   iii. $m(n\vec{A}) = (mn)\vec{A}$

The first two are **Distributive Laws** for *Addition*, and the last one is for *Multiplication*.
As you can see, these only involve the interactions of scalars and vectors. There are other laws too, for vectors only. For that, we need to know the multiplication or product of vectors. For now, just know that there are two types of products possible for a vector: a) the **Dot** (.) or **Scalar product** and b) the **Cross** (×) or **Vector product**. We have **Distributive laws** for these two as well:

iv. $\vec{A}.(\vec{B}+\vec{C}) = \vec{A}.\vec{B}+\vec{A}.\vec{C}$

v. $\vec{A}\times(\vec{B}+\vec{C}) = \vec{A}\times\vec{B}+\vec{A}\times\vec{C}$

For a physical quantity to be called a Vector, it *must* follow these properties, in addition to having both magnitude and direction. As an example, the **electric current** has both direction and magnitude but does not follow these laws, and hence, is a scalar, *not a vector*. **Time** too, is scalar, for the same reason.

3. Vector Addition:

As vectors have direction, adding two or more vectors is more complicated than adding their magnitudes. The ***resultant vector*** $\vec{R}$ obtained after adding two vectors $\vec{P}$ and $\vec{Q}$, can be found by applying one of the three equivalent laws of vector addition:

A. Triangle Law:

When two vectors of the same class are represented as two sides of the triangle with the order of magnitude and direction, then the third side of the triangle represents the magnitude and direction of the resultant vector.

In the adjacent figure, $\vec{P}$ and $\vec{Q}$ are respectively represented by two sides $\overrightarrow{OA}$ and $\overrightarrow{AB}$ of the triangle OAB, in terms of both magnitude and direction. The third side $\overrightarrow{OB}$ then represents the resultant $\vec{R}$ of these vectors, i.e. $\vec{R} = \vec{P}+\vec{Q}$. If the angle between the vectors $\vec{P}$ and $\vec{Q}$ is θ, then

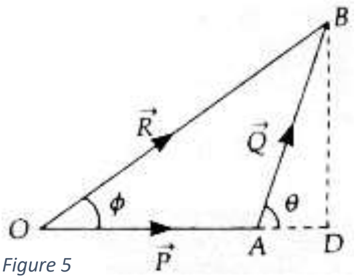

Figure 5

$$|\vec{R}| = R = \sqrt{P^2+Q^2+2\,PQ\cos\theta}$$

In addition, if the angle that $\vec{R}$ makes with $\vec{P}$ is ϕ, then

$$\tan\phi = \frac{Q\sin\theta}{P+Q\cos\theta}$$

B. Parallelogram Law:

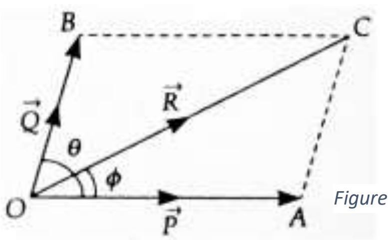

Figure 6

If two vectors of the same class, acting on the same point, are represented by two adjacent sides of a parallelogram, then the diagonal of the parallelogram through the common point represents the sum of the two vectors in both magnitude and direction.

In the adjacent figure, $\vec{P}$ and $\vec{Q}$ are respectively represented by two adjacent sides $\overrightarrow{OA}$ and $\overrightarrow{AB}$ of the parallelogram $OACB$, in terms of both magnitude and direction. The diagonal $\overrightarrow{OC}$ then represents the resultant $\vec{R}$ of these vectors, i.e. $\vec{R} = \vec{P}+\vec{Q}$. If the angle between the vectors $\vec{P}$ and $\vec{Q}$ is θ, then $|\vec{R}| = R = \sqrt{P^2+Q^2+2\,PQ\cos\theta}$. In addition, if the angle that $\vec{R}$ makes with $\vec{P}$ is ϕ, then $\tan\phi = \frac{Q\sin\theta}{P+Q\cos\theta}$

C. Polygon Law:

If $(n-1)$ number of vectors are represented by $(n-1)$ sides of a polygon in sequence, then nth side, closing the polygon in the opposite direction, represents the sum of the vectors in both magnitude and direction.

Let's say, $\vec{a}$, $\vec{b}$, $\vec{c}$, and $\vec{d}$ are four coplanar vectors, depicted by the four sides $\overrightarrow{OA}$, $\overrightarrow{AB}$, $\overrightarrow{BC}$, and $\overrightarrow{CD}$ of the

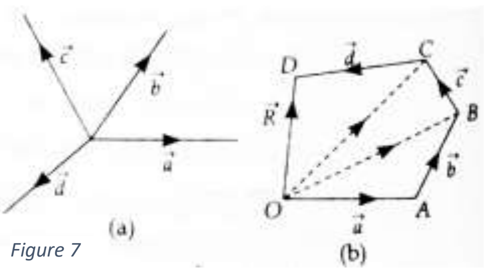

Figure 7

open polygon $OABCD$, in both magnitude and direction. Then the last side of this polygon $\overrightarrow{OD}$, taken in the opposite direction, expresses the resultant $\vec{R}$ of these four vectors in both magnitude and direction. $\therefore\ \vec{R} = \vec{a} + \vec{b} + \vec{c} + \vec{d}$

4. Vector Multiplication

A. Vectors in the Cartesian coordinate system

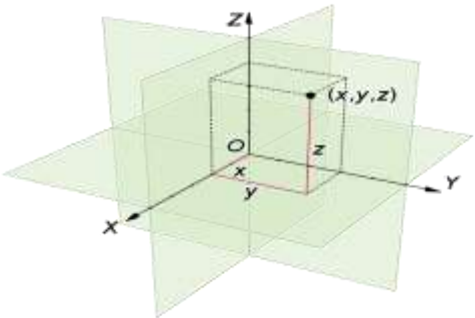

Figure 8
By Jorge Stolfi - Own work, Public Domain, https://commons.wikimedia.org/w/index.php?curid=6692547

A Cartesian coordinate system is a coordinate system that specifies each point uniquely by a set of numerical coordinates, which are the signed distances to the point from three fixed perpendicular oriented lines, measured in the same unit of length. Each reference line is called a coordinate axis or just axis (plural axes) of the system, and the point where they meet is its origin, at ordered pair $(0, 0, 0)$. The coordinates can also be defined as the positions of the perpendicular projections of the point onto the three axes, expressed as signed distances from the origin.

In three dimensions, a Cartesian coordinate system can be of two types, depending on their handedness. In the adjacent figure, the two coordinate systems are called a *left-handed* and a *right-handed* system respectively. In all of our discussions, *we will talk about a right-handed coordinate system*.

To remember, think of a screw which you are rotating from positive x axis to positive y axis, the screw will move towards the positive z direction in a right-handed coordinate system. Another way of remembering is that if we curl the fingers of our right hand in the direction of a $90°$ rotation from the positive x axis to the positive y axis, then the thumb will point to the positive z axis.

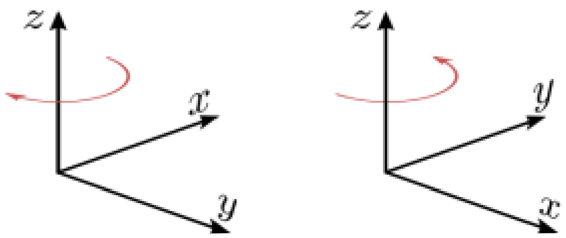

Figure 9
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Remember the definition of **unit vectors** from section *1.B.*? An important set of three unit vectors, called $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ can be defined such that they represent the positive directions of the x, y, and z axes of a Cartesian coordinate system. Any vector $\vec{A}$ in three dimensional Cartesian coordinate system can be represented with an initial point at the origin $\mathcal{O} = (0, 0, 0)$ and its endpoint at some point, say, (A_1, A_2, A_3). Then the vectors $A_1\hat{\imath}$, $A_2\hat{\jmath}$, and $A_3\hat{k}$ are called the ***component vectors*** and A_1, A_2, and A_3 the ***components*** of $\vec{A}$ in the x, y, and z directions, respectively. As any vector in 3-dimension can be expressed with the help of $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$, they are called the **basis vectors** of the Cartesian coordinate system. Hence, $\vec{A} = A_1\hat{\imath} + A_2\hat{\jmath} + A_3\hat{k}$

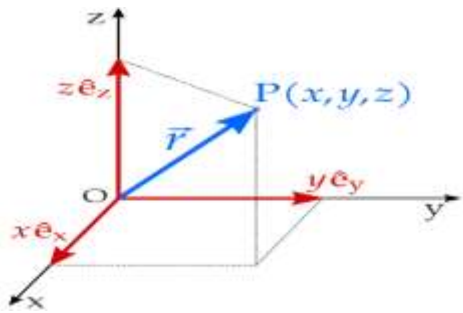

Figure 10
By Original: Jack Ver at Dutch Wikipedia Vector: Ponor - Own work based on: Plaatsvector.png by Jack Ver at Dutch Wikipedia (Original text: Inspired by Plaatsvector.png), CC BY-SA 4.0, https://commons.wikimedia.org/w/index.php?curid=95477901

This is very useful for vector addition and scalar multiplication:

$\vec{A} \pm \vec{B} = (A_1 \pm B_1)\hat{\imath} + (A_2 \pm B_2)\hat{\jmath} + (A_3 \pm B_3)\hat{k}$ and $m\vec{A} = mA_1\hat{\imath} + mA_2\hat{\jmath} + mA_3\hat{k}$.

Using the methodology of the paragraph above, we can assign a vector to each point $P = (x, y, z)$ in space with respect to the origin $O = (0, 0, 0)$ (starting from the origin and ending on the point). This is called the ***Position Vector*** $\overrightarrow{OP} = \vec{r}$ of the point.

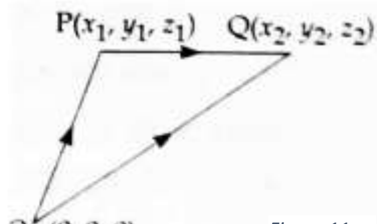

Figure 11

$\therefore\ \vec{r} = x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}\,,$

$|\vec{r}| = \sqrt{x^2 + y^2 + z^2}$

This enables us to measure the distance between two points with the help of vectors. Let's say that the origin is O and the position vectors of two points P and Q are $(x_1\hat{\imath} + y_1\hat{\jmath} + z_1\hat{k})$ and $(x_2\hat{\imath} + y_2\hat{\jmath} + z_2\hat{k})$ respectively. From the adjacent figure, $\overrightarrow{OQ} = \overrightarrow{OP} + \overrightarrow{PQ} \Rightarrow \overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$

$$= (x_2 - x_1)\hat{\imath} + (y_2 - y_1)\hat{\jmath} + (z_2 - z_1)\hat{k}$$
$$\Rightarrow |\overrightarrow{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Here the last line depicts the distance between the two points.

B. Dot Product

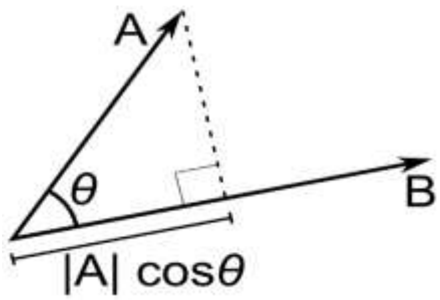

Figure 12 Scalar Projection

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Suppose the magnitudes of two vectors $\vec{A}$ and $\vec{B}$ are A and B and the intermediate angle is θ. Then the Dot product is defined as

$$\boxed{\vec{A}.\vec{B} = AB\cos\theta}$$
$$= BA\cos\theta$$
$$= BA\cos(-\theta) = \vec{B}.\vec{A}$$

The result on the left-hand side is a *scalar*. This is why it is also called a ***Scalar Product***.

Example: when *force* and *displacement* are respectively $\vec{F}$ and $\vec{s}$, then the *work done* by the force $W = \vec{F}.\vec{s}$.

Power$= \frac{dW}{dt} = \frac{d}{dt}(\vec{F}.\vec{s}) = \vec{F}.\frac{d\vec{s}}{dt} = \vec{F}.\vec{v}$, where $\vec{v} = \frac{d\vec{s}}{dt}$ is the velocity.

Let us list some rules of the dot product of vectors:

1. $\vec{A}.\vec{B} = \vec{B}.\vec{A}$ (**Commutative Law**)
2. $\vec{A}.(\vec{B} + \vec{C}) = \vec{A}.\vec{B} + \vec{A}.\vec{C}$ (**Distributive Law**)
3. $m(\vec{A}.\vec{B}) = (m\vec{A}).\vec{B} = \vec{A}.(m\vec{B}) = (\vec{A}.\vec{B})m$ where m is a scalar
4. $\hat{\imath}.\hat{\imath} = \hat{\jmath}.\hat{\jmath} = \hat{k}.\hat{k} = (1)(1)\cos 0° = 1$
5. $\hat{\imath}.\hat{\jmath} = \hat{\jmath}.\hat{k} = \hat{k}.\hat{\imath} = (1)(1)\cos 90° = 0$
6. $\vec{A}.\vec{A} = |\vec{A}|^2\cos 0° = |\vec{A}|^2 = A^2$

   This directly means that the magnitude (***norm***) of a vector $\vec{A}$, $\boxed{A = \sqrt{\vec{A}.\vec{A}}}$
7. If $\vec{A}.\vec{B} = 0$ and none of $\vec{A}$ and $\vec{B}$ are null vectors, then $\vec{A}$ and $\vec{B}$ *are* <u>*perpendicular*</u> *to each other*. They are also called ***orthogonal*** to each other.
8. If two vectors $\vec{A}$ and $\vec{B}$ are expressed in terms of their components in a Cartesian coordinate system, $\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$ and $\vec{B} = B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}$, then

$$\vec{A}.\vec{B} = (A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}).(B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k})$$
$$= A_xB_x(\hat{\imath}.\hat{\imath}) + A_yB_y(\hat{\jmath}.\hat{\jmath}) + A_zB_z(\hat{k}.\hat{k})$$

$$= A_xB_x + A_yB_y + A_zB_z$$

9. If the angle between two vectors $\vec{A}$ and $\vec{B}$, then $\cos\theta = \frac{\vec{A}.\vec{B}}{AB}$
10. (Check Figure 12)

    If the unit vector is in the direction of vector $\vec{B}$ is $\hat{b}$, then the component of the vector $\vec{A}$ in that direction is $\vec{A}.\hat{b} = \frac{\vec{A}.\vec{B}}{B} = A\cos\theta$. So the component vector of $\vec{A}$ in the direction of $\vec{B}$ is

$$(A\cos\theta)\hat{b} = \left(\frac{\vec{A}.\vec{B}}{B}\right)\frac{\vec{B}}{B} = \frac{(\vec{A}.\vec{B})\vec{B}}{B^2}$$

C. Direction Cosines of a Vector

In analytic geometry, the *direction cosines* (or *directional cosines*) of a vector are the cosines of the angles between the vector and the three coordinate axes. Equivalently, they are the contributions of each component of the basis to a unit vector in that direction.

Figure 13
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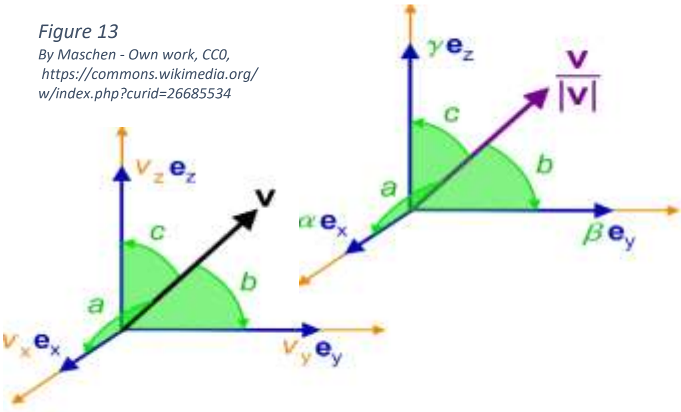


A vector $\vec{V} = V_x\hat{\imath} + V_y\hat{\jmath} + V_z\hat{k}$ makes angles a, b, and c with the three positive axes respectively. Hence the direction cosines are $\alpha = \cos a$, $\beta = \cos b$, and $\gamma = \cos c$. Now,

$$\hat{\imath}.\vec{V} = V\cos a = V\alpha$$
$$\Rightarrow \hat{\imath}.(V_x\hat{\imath} + V_y\hat{\jmath} + V_z\hat{k}) = V_x = V\alpha$$

$\Rightarrow \alpha = \frac{V_x}{V}$.

Similarly, $\beta = \frac{V_y}{V}$, and $\gamma = \frac{V_z}{V}$.

$$\therefore \alpha^2 + \beta^2 + \gamma^2 = \frac{(V_x^2 + V_y^2 + V_z^2)}{V^2} = 1$$

The unit vector in the direction of vector $\vec{V}$,

$$\hat{v} = \frac{\vec{V}}{V} = \frac{V_x}{V}\hat{\imath} + \frac{V_y}{V}\hat{\jmath} + \frac{V_z}{V}\hat{k}$$
$$\Rightarrow \hat{v} = \alpha\hat{\imath} + \beta\hat{\jmath} + \gamma\hat{k}$$

D. Cross Product

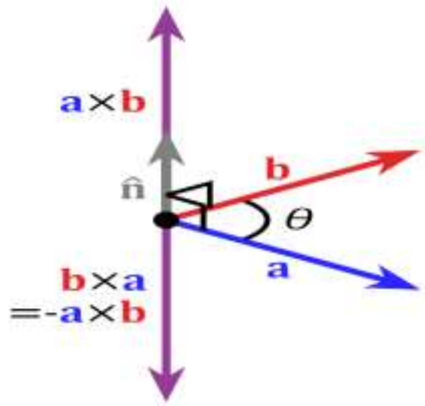


Figure 14
By User:Acdx - Self-made, based on Image:Crossproduct.png, Public Domain, https://commons.wikimedia.org/w/index.php?curid=4436304

Two vectors $\vec{A}$ and $\vec{B}$ with magnitudes A and B has an angle θ between them. The Cross Product between $\vec{A}$ and $\vec{B}$ will be represented as:

$$\vec{C} = \vec{A} \times \vec{B} = (A\,B\sin\theta)\hat{n} \qquad\qquad 0 \le \theta < \pi$$

where $\hat{n}$ is the unit vector in the direction of $\vec{C}$. The result of the cross-product of two vectors is a vector too. This is why it is also called a ***Vector Product***. The direction of $\vec{C}$ is such that it is perpendicular to the plane containing $\vec{A}$ and $\vec{B}$ while $\vec{A}$, $\vec{B}$, and $\vec{C}$ form a right-handed system (i.e. if we curl the fingers of our right hand in the direction of a 90° rotation from the direction of $\vec{A}$ to the direction of $\vec{B}$, then the thumb will point to the direction of $\vec{C}$).

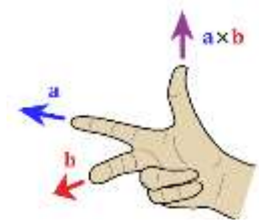


Figure 15
By Acdx - Self-made, based on Image:Right_hand_cross_product.png, CC BY-SA 3.0, https://commons.wikimedia.org/w/index.php?curid=4436743

Let us list some rules of the dot product of vectors:

1. $\vec{A} \times \vec{B} = -(\vec{B} \times \vec{A})$ (Commutative Law **Fails**)
2. $\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$ (**Distributive Law**)
3. $m(\vec{A} \times \vec{B}) = (m\vec{A}) \times \vec{B} = \vec{A} \times (m\vec{B}) = (\vec{A} \times \vec{B})m$ where m is a scalar
4. $\vec{A} \times \vec{A} = |\vec{A}|^2 \sin 0° = 0$
5. If $\vec{A} \times \vec{B} = 0$ and none of $\vec{A}$ and $\vec{B}$ are null vectors, then $\vec{A}$ and $\vec{B}$ *are **parallel** to each other.*
6. If $\vec{A}$ and $\vec{B}$ are perpendicular to each other, then $|\vec{A} \times \vec{B}| = A\,B$.
7. If the angle between two vectors $\vec{A}$ and $\vec{B}$, then $\sin\theta = \frac{|\vec{A}\times\vec{B}|}{A\,B}$.
8. ***Significance of Cross Product***: *The magnitude of $\vec{A} \times \vec{B}$ is the same as the area of a parallelogram with sides $\vec{A}$ and $\vec{B}$.*

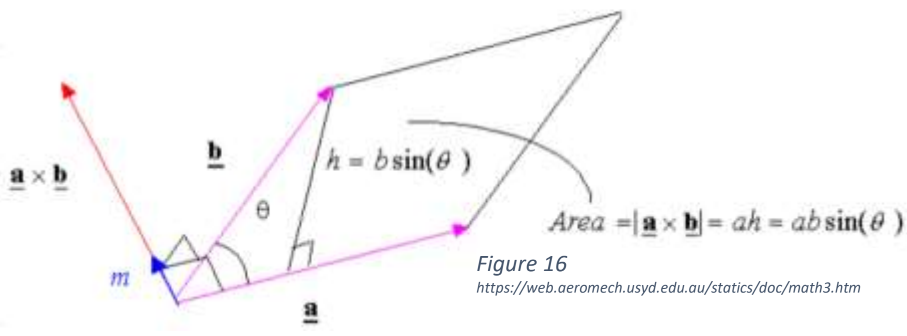


Figure 16
https://web.aeromech.usyd.edu.au/statics/doc/math3.htm

Let's say that $\vec{a}$ and $\vec{b}$ are represented by two adjacent sides $\overline{OP}$ and $\overline{OR}$ of a parallelogram $OPQR$. Now the area of $OPQR$, from trigonometry, is

$\Delta = \frac{1}{2}\,OP\,h = \frac{1}{2}(OP)(OR\sin\theta) \qquad [\because \frac{h}{OR} = \sin\theta]$

$= \frac{1}{2}a\,b\sin\theta = \frac{1}{2}|\vec{a} \times \vec{b}|$.

$\therefore (\Delta)\hat{m} = \frac{1}{2}|\vec{a} \times \vec{b}|\hat{m}$, where $\hat{m}$ is the unit vector perpendicular to the plane containing $\vec{a}$ and $\vec{b}$.

$\therefore \vec{\Delta} = \frac{1}{2}(\vec{a} \times \vec{b})$

$\Rightarrow \vec{a} \times \vec{b} = 2\vec{\Delta} =$ area of the parallelogram $OPQR$.

9. ***Cross Product in Cartesian Coordiante system***:

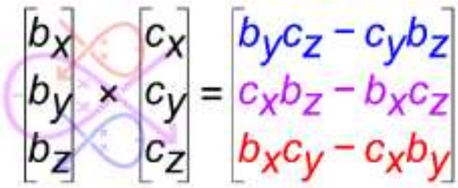


Figure 17
By Cmglee - Own work, CC BY-SA 4.0, https://commons.wikimedia.org/w/index.php?curid=93694405

$\hat{\imath} \times \hat{\imath} = \hat{\jmath} \times \hat{\jmath} = \hat{k} \times \hat{k} = (1)(1)\sin 0° = 0$

$\hat{\imath} \times \hat{\jmath} = (1)(1)\sin 90° = \hat{n}$; here $\hat{n}$ is perpendicular to both $\hat{\imath}$ and $\hat{\jmath}$ and its direction is determined by the right-hand rule.

$\therefore \hat{\imath} \times \hat{\jmath} = \hat{k}$ and $\hat{\jmath} \times \hat{\imath} = -\hat{k}$

Similarly, $\hat{\jmath} \times \hat{k} = \hat{\imath}$; $\hat{k} \times \hat{\imath} = \hat{\jmath}$.

Cross product of any two vectors can be specified very simply in the Cartesian coordinate system. If $\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$ and $\vec{B} = B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}$, then

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$= \begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix} \hat{\imath} + \begin{vmatrix} A_z & A_x \\ B_z & B_x \end{vmatrix} \hat{\jmath} + \begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix} \hat{k}$$

$$= (A_yB_z - A_zB_y)\hat{\imath} + (A_zB_x - A_xB_z)\hat{\imath} + (A_xB_y - A_yB_x)\hat{k}$$

$$= (A_yB_z - A_zB_y)\hat{\imath} - (A_xB_z - A_zB_x)\hat{\imath} + (A_xB_y - A_yB_x)\hat{k}$$

E. Triple Product of Vectors

Dot and cross multiplication of three vectors give rise to interesting products of vectors, called ***Triple Products***. These are of two types: *Scalar Triple Product* and *Vector Triple Product*.

When the multiplication of three vectors gives rise to a scalar quantity, it is called a **scalar triple product**. Example: $\vec{A}.(\vec{B} \times \vec{C})$.

Let's say the three vectors are expressed in terms of their Cartesian components: $\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$, $\vec{B} = B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}$, and $\vec{C} = C_x\hat{\imath} + C_y\hat{\jmath} + C_z\hat{k}$.

$$\vec{B} \times \vec{C} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} = (B_yC_z - B_zC_y)\hat{\imath} + (B_zC_x - B_xC_z)\hat{\imath} + (B_xC_y - B_yC_x)\hat{k}$$

$$\therefore \vec{A}.(\vec{B} \times \vec{C}) = (A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}).\{(B_yC_z - B_zC_y)\hat{\imath} + (B_zC_x - B_xC_z)\hat{\imath} + (B_xC_y - B_yC_x)\hat{k}\}$$

$$= A_x(B_yC_z - B_zC_y) + A_y(B_zC_x - B_xC_z) + A_z(B_xC_y - B_yC_x)$$

$$= \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} = -\begin{vmatrix} B_x & B_y & B_z \\ A_x & A_y & A_z \\ C_x & C_y & C_z \end{vmatrix} \qquad \text{[following the rules of Determinant]}$$

$$= \begin{vmatrix} B_x & B_y & B_z \\ C_x & C_y & C_z \\ A_x & A_y & A_z \end{vmatrix} = \vec{B}.(\vec{C} \times \vec{A})$$

$$= -\begin{vmatrix} C_x & C_y & C_z \\ B_x & B_y & B_z \\ A_x & A_y & A_z \end{vmatrix}$$

$$= \begin{vmatrix} C_x & C_y & C_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \vec{C}.(\vec{A} \times \vec{B})$$

$$\therefore \vec{A}.(\vec{B} \times \vec{C}) = \vec{B}.(\vec{C} \times \vec{A}) = \vec{C}.(\vec{A} \times \vec{B})$$

The Geometric Significance of the scalar triple product: The scalar triple product denotes the volume of a parallelepiped if three adjacent edges of it represent the three vectors.

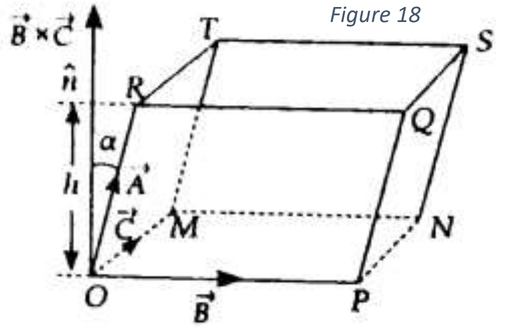


Figure 18

Let's say $\overrightarrow{OP}$, $\overrightarrow{OM}$, and $\overrightarrow{OR}$ represents the three vectors $\vec{B}$, $\vec{C}$, and $\vec{A}$ both in magnitude and direction. As we have learned earlier, the area of the base (enclosed by $\vec{B}$ and $\vec{C}$) of the parallelepiped is $\vec{B}\times\vec{C}$, where the direction of it is perpendicular to the plane containing $\vec{B}$ and $\vec{C}$, i.e. the base and is denoted by the unit vector $\hat{n}$. Say the angle that $\hat{n}$ makes with $\vec{A}$ is α. If the height of the parallelepiped is h, then $\frac{h}{A}=\cos\alpha$

$\Rightarrow h = A\cos\alpha$

$\therefore$ the volume of the parallelepiped is

$$V = h|\vec{B}\times\vec{C}| = A\,|\vec{B}\times\vec{C}|\cos\alpha = \vec{A}.(\vec{B}\times\vec{C})$$

Corollary: If $\vec{A}.(\vec{B}\times\vec{C})=0$, and none of the vectors are null, then the volume of the parallelepiped made by them is zero, i.e. the three vectors are coplanar.

Vector triple products are of two types. For three vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$, these are respectively $(\vec{A}.\vec{B})\vec{C}$ and $\vec{A}\times(\vec{B}\times\vec{C})$.

In general, $(\vec{A}.\vec{B})\vec{C} \neq \vec{A}(\vec{B}.\vec{C})$ and $\vec{A}\times(\vec{B}\times\vec{C}) \neq (\vec{A}\times\vec{B})\times\vec{C}$

$$\vec{A}\times(\vec{B}\times\vec{C}) = (\vec{A}.\vec{C})\vec{B} - (\vec{A}.\vec{B})\vec{C}$$
$$(\vec{A}\times\vec{B})\times\vec{C} = (\vec{A}.\vec{C})\vec{B} - (\vec{B}.\vec{C})\vec{A}$$

5. Solved Questions

1. ***If $\vec{A} = \hat{\imath} - 2\hat{\jmath} + 3\hat{k}$, and $\vec{B} = 2\hat{\imath} + 5\hat{\jmath} - 2\hat{k}$, then determine the magnitude and direction of $\vec{A}+\vec{B}$.***

   Answer: If resultant of $\vec{A}$ and $\vec{B}$ is $\vec{R}$, then

   $\vec{R} = \vec{A}+\vec{B} = (\hat{\imath} - 2\hat{\jmath} + 3\hat{k}) + (2\hat{\imath} + 5\hat{\jmath} - 2\hat{k}) = 3\hat{\imath} + 3\hat{\jmath} + \hat{k}$.

   $\therefore$ The norm of $\vec{R}$, $R = \sqrt{3^2+3^2+1^2} = \sqrt{19}$.

   The unit vector in the direction of $\vec{R}$ is,

   $$\hat{R} = \frac{\vec{R}}{R} = \frac{3\hat{\imath}+3\hat{\jmath}+\hat{k}}{\sqrt{19}} = \frac{3}{\sqrt{19}}\hat{\imath} + \frac{3}{\sqrt{19}}\hat{\jmath} + \frac{1}{\sqrt{19}}\hat{k}$$

2. ***The magnitudes of three vectors in some arbitrary unit are 2, 3, and 6, respectively. Do they satisfy the Triangle Law of vector addition? Justify.***

   Answer: No, these vectors do not satisfy the Triangle law of vector addition, as they do not constitute a triangle. We know that the sum of any two sides of a triangle is larger than the third. As $2+3<6$, the three supplied vectors do not form a triangle.

3. ***Prove the polygon rule of vector addition from the triangle law of vector addition.***

Answer: (Refer to Figure 7)

Let's join OB and OC. Now $\overrightarrow{OA}$ and $\overrightarrow{AB}$ two sequential vectors of the triangle ΔOAB. Hence, following the triangle law of addition, $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$.

Similarly, $\overrightarrow{OC} = \overrightarrow{OB} + \overrightarrow{BC} = \overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BC}$.

Again, $\overrightarrow{OD} = \overrightarrow{OC} + \overrightarrow{CD} = \overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD}$, from the triangle ΔOCD.

$\therefore\ \vec{R} = \vec{a} + \vec{b} + \vec{c} + \vec{d}$ (***QED***)

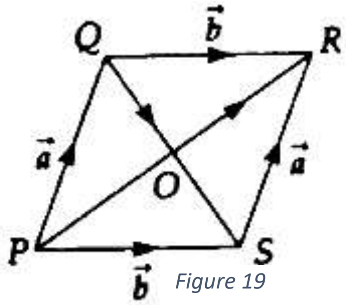


Figure 19

4. ***$\vec{A} = 2\hat{\imath} - 2\hat{\jmath} + \hat{k}$ and $\vec{B} = \hat{\imath} + 2\hat{\jmath} + 3\hat{k}$. Find the component of $\vec{B}$ in the direction of $\vec{A}$.***

Answer: The component of $\vec{B}$ in the direction of $\vec{A}$ is

$$\vec{B}.\hat{a} = \frac{\vec{B}.\vec{A}}{A} = \frac{(\hat{\imath} + 2\hat{\jmath} + 3\hat{k}).(2\hat{\imath} - 2\hat{\jmath} + \hat{k})}{\sqrt{2^2 + 2^2 + 1}} = \frac{2 - 4 + 3}{3} = \frac{1}{3}$$

5. ***If vectors $\vec{A} = 3\hat{\imath} + 4\hat{\jmath} + 5\hat{k}$ and $\vec{B} = p\hat{\imath} - 3\hat{\jmath} + 3\hat{k}$ are orthogonal/perpendicular to each other, then what is the value of p?***

Answer: $\vec{A}$ and $\vec{B}$ are perpendicular to each other if $\vec{A}.\vec{B} = 0$.

$$\therefore (3\hat{\imath} + 4\hat{\jmath} + 5\hat{k}).(p\hat{\imath} - 3\hat{\jmath} + 3\hat{k}) = 0$$
$$\Rightarrow 3p - 12 + 15 = 0$$
$$\Rightarrow 3p = -3$$
$$\Rightarrow p = -1$$

6. ***If $\vec{A} = 3\hat{\imath} + 5\hat{\jmath} - 2\hat{k}$ and $\vec{B} = 5\hat{\imath} - 2\hat{\jmath}$, then find the projection of $\vec{A}$ on $\vec{B}$.*** [VU 2018]

Answer: If $\hat{b}$ is the unit vector in the direction of $\vec{B}$, then the projection of $\vec{A}$ in that direction,

$$\vec{A}.\hat{b} = (3\hat{\imath} + 5\hat{\jmath} - 2\hat{k}).\frac{(5\,\hat{\imath} - 2\,\hat{\jmath})}{\sqrt{5^2 + 2^2}}$$
$$= \frac{15 - 10}{\sqrt{29}} = \frac{5}{\sqrt{29}}$$

7. ***Find the direction cosines of the vector $\vec{A} = \hat{\imath} + 2\hat{\jmath} + 2\hat{k}$.***

Answer: Here, the components of the vector are $A_x = 1, A_y = 2$, and $A_z = 2$. So the norm of the vector, $A = \sqrt{A_x^2 + A_y^2 + A_z^2} = \sqrt{1 + 4 + 4} = 3$. Hence, the direction cosines are,

$$\left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$$

8. ***The norm/magnitude of a vector is 5 and the direction cosines are respectively $\frac{1}{2}, \frac{1}{\sqrt{2}}$, and $\frac{1}{2}$ (in some arbitrary unit). Find the vector.***

Answer:

$A = 5$. The direction cosines,

$$(\alpha, \beta, \gamma) = \left(\frac{A_x}{A}, \frac{A_y}{A}, \frac{A_z}{A}\right) = \left(\frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{1}{2}\right)$$
$$\Rightarrow (A_x, A_y, A_z) = \left(\frac{5}{2}, \frac{5}{\sqrt{2}}, \frac{5}{2}\right)$$

Hence, the vector $\vec{A} = \frac{5}{2}\hat{\imath} + \frac{5}{\sqrt{2}}\hat{\jmath} + \frac{5}{2}\hat{k}$.

9. ***Show that the diagonals of a parallelogram bisect each other.***

Answer:

$\overrightarrow{PR}$ and $\overrightarrow{QS}$, diagonals of a parallelogram $PQRS$, cross each other at the point O. Let's say

$\overrightarrow{PO} = m\,\overrightarrow{PR}$ and $\overrightarrow{QO} = n\,\overrightarrow{QS}$. Now, using the law of vector addition, we can write,

$$\overrightarrow{PQ} + \overrightarrow{QS} = \overrightarrow{PS}$$
$$\therefore \overrightarrow{QS} = \overrightarrow{PS} - \overrightarrow{PQ} = \vec{b} - \vec{a}$$
$$\therefore \overrightarrow{QO} = n\,\overrightarrow{QS} = n(\vec{b} - \vec{a})$$

On the other hand,

$\overrightarrow{PS} + \overrightarrow{SR} = \overrightarrow{PR}$ and $\overrightarrow{PR} = \vec{b} + \vec{a}$

$$\therefore \overrightarrow{PO} = m\,\overrightarrow{PR} = m(\vec{b} + \vec{a})$$

We can write $\overrightarrow{PQ} + \overrightarrow{QO} = \overrightarrow{PO}$ for the triangle ΔPQO.

$$\therefore \overrightarrow{PQ} = \overrightarrow{PO} - \overrightarrow{QO} = m(\vec{b} + \vec{a}) - n(\vec{b} - \vec{a})$$
$$\Rightarrow \vec{a} = (m - n)\vec{b} + (m + n)\vec{a}$$

As $\vec{a}$ and $\vec{b}$ do not lie on the same straight line, this is only true when

$(m - n) = 0 \Rightarrow m = n$ and $(m + n) = 1$

$$\therefore m = n = \frac{1}{2}$$

Hence, diagonals of a parallelogram bisect each other. (***QED***)

10. ***If $\vec{A} = 3\,\hat{\imath} - 2\,\hat{\jmath} + \hat{k}$ and $\vec{A} = 3\,\hat{\imath} - 2\,\hat{\jmath} + \hat{k}$, then find the angle between them.***
[CU 2016, 2011, BU 2016]

Answer:

We know that $\vec{A}.\vec{B} = A\,B\cos\theta$.

$$\therefore \cos\theta = \frac{\vec{A}.\vec{B}}{A\,B}$$

Now, $\vec{A}.\vec{B} = (3\hat{\imath} + 2\hat{\jmath} - 6\hat{k}).(4\hat{\imath} - 3\hat{\jmath} + \hat{k}) = 12 - 6 - 6 = 0$

$$\Rightarrow \cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$$

The angle between the vectors is 90° i.e. those vectors are perpendicular to each other.

11. ***For what value of x are the two vectors $\vec{A} = \hat{\imath} + x\hat{\jmath} + \hat{k}$ and $\vec{B} = 3\hat{\imath} - 2\hat{\jmath} - 2\hat{k}$ perpendicular to each other?*** ***[CU 2015, 2013]***

Answer:

The condition for $\vec{A}$ and $\vec{B}$ to be perpendicular is $\vec{A}.\vec{B} = 0$.

$$\Rightarrow (\hat{\imath} + x\hat{\jmath} + \hat{k}).(3\hat{\imath} - 2\hat{\jmath} - 2\hat{k}) = 0$$
$$\Rightarrow 3 - 2x - 2 = 0$$
$$\Rightarrow x = \frac{1}{2}$$

12. ***Find the unit vector in the direction of the vector $\vec{A} = 3\hat{\imath} + 4\hat{\jmath} + \hat{k}$.*** ***[CU 2014]***

Answer:

$$\vec{A} = 3\hat{\imath} + 4\hat{\jmath} + \hat{k}$$
$$\Rightarrow |\vec{A}| = \sqrt{3^2 + 4^2 + 1^2} = \sqrt{26}$$

Hence, the unit vector in the direction of $\vec{A}$:

$$\hat{a} = \frac{\vec{A}}{|\vec{A}|} = \frac{3\hat{\imath} + 4\hat{\jmath} + \hat{k}}{\sqrt{26}}$$

13. ***If $|\vec{A} + \vec{B}| = |\vec{A} - \vec{B}|$, then prove that $\vec{A}$ and $\vec{B}$ are perpendicular to each other.***
[CU 2012, BU 2017]

Answer:

$$|\vec{A} + \vec{B}| = |\vec{A} - \vec{B}|$$
$$\Rightarrow A^2 + B^2 + 2AB\cos\theta = A^2 + B^2 - 2AB\cos\theta$$

$$\Rightarrow \cos\theta = 0 \Rightarrow \theta = 90°$$

14. ***If $\vec{A}$ is a vector with constant magnitude, then prove that $\frac{d\vec{A}}{dt}$ and $\vec{A}$ are perpendicular to each other.*** ***[CU 2011]***

Answer:

$\vec{A}$ is a vector with constant magnitude, i.e. $|\vec{A}|$ =constant.$\Rightarrow \frac{d}{dt}|\vec{A}| = 0$

$$\vec{A}.\vec{A} = |\vec{A}|^2$$

$$\Rightarrow 2\vec{A}.\frac{d\vec{A}}{dt} = 2|\vec{A}|\frac{d}{dt}|\vec{A}| \quad \text{[Differentiating with respect to } t\text{]}$$

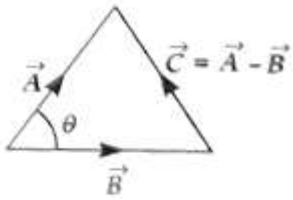
$$\Rightarrow \vec{A}.\frac{d\vec{A}}{dt} = 0 \qquad \left[\because \frac{d}{dt}|\vec{A}| = 0\right]$$

$\therefore \frac{d}{dt}|\vec{A}|$ and $\vec{A}$ are perpendicular to each other.

15. ***Show that $C^2 = A^2 + B^2 - 2AB\cos\theta$ for a triangle.*** ***[VU 2018]***

Answer:

From the adjacent figure,



$$C^2 = \vec{C}.\vec{C}$$

$$\Rightarrow C^2 = (\vec{A} - \vec{B}).(\vec{A} - \vec{B})$$

$$= A^2 + B^2 - 2\vec{A}.\vec{B}$$

$$= A^2 + B^2 - 2\,A\,B\cos\theta$$

16. ***If $\vec{A}\times\vec{B} + \vec{B}\times\vec{C} + \vec{C}\times\vec{A} = 0$, then determine whether the three vectors are coplanar or not.***

Answer:

$$\vec{A}\times\vec{B} + \vec{B}\times\vec{C} + \vec{C}\times\vec{A} = 0$$

$$\Rightarrow \vec{A}.\left(\vec{A}\times\vec{B} + \vec{B}\times\vec{C} + \vec{C}\times\vec{A}\right) = 0$$

$$\Rightarrow \vec{A}.\left(\vec{A}\times\vec{B}\right) + \vec{A}.\left(\vec{B}\times\vec{C}\right) + \vec{A}.\left(\vec{C}\times\vec{A}\right) = 0$$

Now, both $\left(\vec{A}\times\vec{B}\right)$ and $\left(\vec{C}\times\vec{A}\right)$ are perpendicular to $\vec{A}$.

$$\therefore \vec{A}.\left(\vec{A}\times\vec{B}\right) = \vec{A}.\left(\vec{C}\times\vec{A}\right) = 0$$

$$\Rightarrow \vec{A}.\left(\vec{B}\times\vec{C}\right) = 0$$

Hence, the three vectors are coplanar.

17. ***Three vectors are: $\vec{A} = 3\,\hat{\imath} - 2\,\hat{\jmath} + \hat{k}$, $\vec{B} = \hat{\imath} + \hat{\jmath} - 2\,\hat{k}$, and $\vec{C} = 3\,\hat{\imath} - 4\,\hat{\jmath} + \lambda\,\hat{k}$. What is the value of λ so that the three vectors are coplanar?***

Answer: The relation that ensures that $\vec{A}$, $\vec{B}$, and $\vec{C}$ are coplanar is $\vec{A}.\left(\vec{B}\times\vec{C}\right) = 0$

$$\vec{A}.\left(\vec{B}\times\vec{C}\right) = \begin{vmatrix} 3 & -2 & 1 \\ 1 & 1 & -2 \\ 3 & -4 & \lambda \end{vmatrix} = 3(\lambda - 8) + 2(\lambda + 6) + 1(-4 - 3)$$

$$= 3\lambda - 24 + 2\lambda + 12 - 7 = 5\lambda - 19$$

Hence for the vectors to be coplanar, $\lambda = \frac{19}{5}$.

18. ***Show that $(\vec{\omega}\times\vec{r})^2 = \vec{\omega}.\{\vec{r}\times(\vec{\omega}\times\vec{r})\}$***

Answer:

$(\vec{\omega}\times\vec{r})^2 = (\vec{\omega}\times\vec{r}).(\vec{\omega}\times\vec{r})$

$= \vec{\omega}.\{\vec{r}\times(\vec{\omega}\times\vec{r})\}$ $\quad \left[\because \vec{A}.\left(\vec{B}\times\vec{C}\right) = \vec{B}.\left(\vec{C}\times\vec{A}\right)\right]$

19. ***$\vec{A} = 2\,\hat{\imath} - 2\,\hat{\jmath} + \hat{k}$ and $\vec{B} = \hat{\imath} + 2\,\hat{\jmath} + 3\,\hat{k}$. Find the component of $\vec{B}$ in the direction of $\vec{A}$.***

Answer:

The unit vector in the direction of $\vec{A}$ is $\hat{a} = \frac{\vec{A}}{|\vec{A}|} = \frac{(2\,\hat{\imath} - 2\,\hat{\jmath} + \hat{k})}{\sqrt{4+4+1}} = \frac{2}{3}\,\hat{\imath} - \frac{2}{3}\,\hat{\jmath} + \frac{1}{3}\,\hat{k}$.

$\therefore$ the component of $\vec{B}$ in the direction of $\vec{A} = \vec{B}.\hat{a} = (\hat{\imath} + 2\hat{\jmath} + 3\hat{k}).\left(\frac{2}{3}\hat{\imath} - \frac{2}{3}\hat{\jmath} + \frac{1}{3}\hat{k}\right)$
$= \frac{(2-4+3)}{3} = \frac{1}{3}$.

20. ***$\vec{A} = \hat{\imath} + 2\hat{\jmath} + 3\hat{k}$ and $\vec{B} = 2\hat{\imath} - \hat{\jmath} + 2\hat{k}$. Find the unit vector perpendicular to both $\vec{A}$ and $\vec{B}$.***

Answer:

We know that the $(\vec{A} \times \vec{B})$ vector is perpendicular to both $\vec{A}$ and $\vec{B}$. Hence the unit vector in that direction is

$$\hat{n} = \pm\frac{(\vec{A} \times \vec{B})}{|\vec{A} \times \vec{B}|}$$

Now, $\vec{A} \times \vec{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2 & 3 \\ 2 & -1 & 2 \end{vmatrix} = \hat{\imath}(4+3) - \hat{\jmath}(2-6) - \hat{k}(-1-4) = 7\hat{\imath} + 4\hat{\jmath} - 5\hat{k}.$

$$\therefore \hat{n} = \pm\frac{(7\hat{\imath} + 4\hat{\jmath} - 5\hat{k})}{\sqrt{49+16+25}} = \pm\frac{(7\hat{\imath} + 4\hat{\jmath} - 5\hat{k})}{\sqrt{90}}$$

21. ***Show that the vectors $\vec{A} = \hat{\imath} - 2\hat{\jmath} + \hat{k}$ and $\vec{B} = -2\hat{\imath} + 4\hat{\jmath} - 2\hat{k}$ are parallel to each other.***

Answer:

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -2 & 1 \\ -2 & 4 & -2 \end{vmatrix} = \hat{\imath}(4-4) - \hat{\jmath}(-2+2) - \hat{k}(4-4) = \vec{0}$$

Hence they are parallel.

22. ***Show that $\vec{A} \times (\vec{B} \times \vec{C}) + \vec{B} \times (\vec{C} \times \vec{A}) + \vec{C} \times (\vec{A} \times \vec{B}) = \vec{0}$.*** [CU 2003, BU 2017]

Answer:

Following the vector triple product result,

$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A}.\vec{C}) - \vec{C}(\vec{A}.\vec{B})$,

$$\Rightarrow \vec{A} \times (\vec{B} \times \vec{C}) + \vec{B} \times (\vec{C} \times \vec{A}) + \vec{C} \times (\vec{A} \times \vec{B})$$
$$= \vec{B}(\vec{A}.\vec{C}) - \vec{C}(\vec{A}.\vec{B}) + \vec{C}(\vec{B}.\vec{A}) - \vec{A}(\vec{B}.\vec{C}) + \vec{A}(\vec{C}.\vec{B}) - \vec{B}(\vec{C}.\vec{A})$$
$$= 0$$

23. ***It is given that $\vec{A} + \vec{B} + \vec{C} = 0$. Show that $\vec{A} \times \vec{B} = \vec{B} \times \vec{C} = \vec{C} \times \vec{A}$*** [VU 2018]

Answer:

$$\vec{A} + \vec{B} + \vec{C} = 0 \Rightarrow \vec{A} + \vec{B} = -\vec{C}$$
$$\Rightarrow \vec{C} \times (\vec{A} + \vec{B}) = -\vec{C} \times \vec{C} = 0$$
$$\Rightarrow \vec{C} \times \vec{A} + \vec{C} \times \vec{B} = 0$$
$$\Rightarrow \vec{C} \times \vec{A} = -\vec{C} \times \vec{B} = \vec{B} \times \vec{C}$$

Similarly,

$$\vec{A} + \vec{B} + \vec{C} = 0 \Rightarrow \vec{A} + \vec{C} = -\vec{B}$$
$$\Rightarrow \vec{A} \times (\vec{A} + \vec{C}) = -\vec{A} \times \vec{B}$$
$$\Rightarrow \vec{A} \times \vec{A} + \vec{A} \times \vec{C} = -\vec{A} \times \vec{B}$$
$$\Rightarrow \vec{C} \times \vec{A} = \vec{A} \times \vec{B} \qquad [\because \vec{A} \times \vec{A} = 0]$$

$\therefore \vec{A} \times \vec{B} = \vec{B} \times \vec{C} = \vec{C} \times \vec{A}$ (**QED**)

24. ***Using vector algebra, show that $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$.*** [CU 2018]

Answer:

For the triangle in figure 19, the angle between $\vec{a}$ and $\vec{c}$ is B, between $\vec{b}$ and $\vec{c}$ is A, and between $\vec{a}$ and $\vec{b}$ is $(\pi - C)$. For the same figure,

$\vec{a} + \vec{b} = \vec{c}$

$$\Rightarrow \vec{b} \times (\vec{a} + \vec{b}) = \vec{b} \times \vec{c}$$
$$\Rightarrow \vec{b} \times \vec{a} = \vec{b} \times \vec{c} \qquad [\because \vec{b} \times \vec{b} = 0]$$
$$\Rightarrow b\, a \sin(\pi - C) = b\, c \sin A$$
$$\Rightarrow a \sin C = c \sin A$$
$$\Rightarrow \frac{a}{\sin A} = \frac{c}{\sin C}$$

Similarly, we can prove $\frac{b}{\sin B} = \frac{c}{\sin C}$, starting from $\vec{a} \times (\vec{a} + \vec{b}) = \vec{a} \times \vec{c}$. Hence,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

25. $(\vec{A} + \vec{B}).\{(\vec{B} + \vec{C}) \times (\vec{C} + \vec{A})\} = ?$

Answer:

$$(\vec{B} + \vec{C}) \times (\vec{C} + \vec{A}) = (\vec{B} \times \vec{C}) + (\vec{B} \times \vec{A}) + (\vec{C} \times \vec{A}) \qquad [\because \vec{C} \times \vec{C} = 0]$$
$$\therefore (\vec{A} + \vec{B}).\{(\vec{B} + \vec{C}) \times (\vec{C} + \vec{A})\} = (\vec{A} + \vec{B}).\{(\vec{B} \times \vec{C}) + (\vec{B} \times \vec{A}) + (\vec{C} \times \vec{A})\}$$
$$= \vec{A}.(\vec{B} \times \vec{C}) + \vec{A}.(\vec{B} \times \vec{A}) + \vec{A}.(\vec{C} \times \vec{A}) + \vec{B}.(\vec{B} \times \vec{C}) + \vec{B}.(\vec{B} \times \vec{A}) + \vec{B}.(\vec{C} \times \vec{A})$$
$$= \vec{A}.(\vec{B} \times \vec{C}) + \vec{B}.(\vec{C} \times \vec{A}) \qquad [\because \vec{A}.(\vec{B} \times \vec{A}) = \vec{A}.(\vec{C} \times \vec{A}) = \vec{B}.(\vec{B} \times \vec{C}) = \vec{B}.(\vec{B} \times \vec{A}) = 0]$$
$$= 2\vec{A}.(\vec{B} \times \vec{C}) \qquad [\because \vec{B}.(\vec{C} \times \vec{A}) = \vec{A}.(\vec{B} \times \vec{C})]$$

26. ***It is given that $\vec{r} \times \vec{b} = \vec{c} \times \vec{b}$ and $\vec{r}.\vec{a} = 0$; $\vec{a}.\vec{b} \neq 0$. Find out $\vec{r}$ in terms of $\vec{a}$, $\vec{b}$, and $\vec{c}$.***

Answer:

It's given that, $\vec{r} \times \vec{b} = \vec{c} \times \vec{b}$

$$\Rightarrow \vec{a} \times (\vec{r} \times \vec{b}) = \vec{a} \times (\vec{c} \times \vec{b})$$
$$\Rightarrow \vec{r}(\vec{a}.\vec{b}) - \vec{b}(\vec{a}.\vec{r}) = \vec{a} \times (\vec{c} \times \vec{b})$$
$$\Rightarrow \vec{r}(\vec{a}.\vec{b}) = \vec{a} \times (\vec{c} \times \vec{b}) \qquad [\because (\vec{r}.\vec{a}) = 0]$$
$$\Rightarrow \vec{r} = \frac{\vec{a} \times (\vec{c} \times \vec{b})}{(\vec{a}.\vec{b})}$$

27. ***A rigid body is rotating with 5 unit angular velocity around an axis parallel to the vector $(4\hat{j} - 2\hat{k})$. The axis passes through the point $(\hat{\imath} + 2\hat{j} - 3\hat{k})$. What would be the velocity of a particle on the rigid body at the point $(3\hat{\imath} - 2\hat{j} + \hat{k})$?*** [BU]

Answer:

Angular velocity $|\vec{\omega}| = 5$ units. Unit vector in the direction of the rotation axis:

$$\hat{n} = \frac{4\hat{j} - 3\hat{k}}{\sqrt{16 + 9}} = \frac{1}{5}(4\hat{j} - 3\hat{k})$$
$$\therefore \vec{\omega} = \omega\hat{n} = 4\hat{j} - 3\hat{k}$$

Now the direction vector from $(\hat{\imath} + 2\hat{j} - 3\hat{k})$ to $(3\hat{\imath} - 2\hat{j} + \hat{k})$ is:

$$\vec{r} = (3\hat{\imath} - 2\hat{j} + \hat{k}) - (\hat{\imath} + 2\hat{j} - 3\hat{k}) = (2\hat{\imath} - 4\hat{j} + 4\hat{k})$$

Hence, the velocity of the particle at $(3\hat{\imath} - 2\hat{j} + \hat{k})$ is:

$$\vec{v} = \vec{\omega} \times \vec{r} = \begin{vmatrix} \hat{\imath} & \hat{j} & \hat{k} \\ 0 & 4 & -3 \\ 2 & -4 & 4 \end{vmatrix} = 4\hat{\imath} - 6\hat{j} + 8\hat{k} \text{ units.}$$

28. ***The force $\vec{F} = 3\hat{\imath} - 2\hat{j} - 4\hat{k}$, acts on the point $(1, -1, 2)$. Find the moment of the force with respect to the point $(2, -1, 3)$.*** [VU 2018]

Answer:

Let's say the vector from the point $A = (2, -1, 3)$ to the point $B = (1, -1, 2)$ is

$\vec{r} = (\hat{\imath} - \hat{j} + 2\hat{k}) - (2\hat{\imath} - \hat{j} + 3\hat{k}) = -\hat{\imath} - \hat{k}$.

The moment $\vec{\tau} = (\vec{r} \times \vec{F}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -1 & 0 & -1 \\ 3 & -2 & -4 \end{vmatrix} = -2\hat{\imath} - 7\hat{\jmath} + 2\hat{k}$

$\therefore |\vec{\tau}| = \sqrt{(-2)^2 + (-7)^2 + 2^2} = \sqrt{57}$ units.

29. ***Find the unit vector $(\hat{\boldsymbol{n}})$ in the direction perpendicular to the plane containing the vectors $\vec{\boldsymbol{A}} = 3\hat{\boldsymbol{\imath}} - 2\hat{\boldsymbol{\jmath}} + 4\hat{\boldsymbol{k}}$ and $\vec{\boldsymbol{B}} = \hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} - 2\hat{\boldsymbol{k}}$.*** *[VU 2018]*

Answer:

$$\vec{n} = \pm \frac{\vec{A} \times \vec{B}}{|\vec{A} \times \vec{B}|}$$

$\therefore \vec{A} \times \vec{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & -2 & 4 \\ 1 & 1 & -2 \end{vmatrix} = 10\hat{\jmath} + 5\hat{k} \Rightarrow |\vec{A} \times \vec{B}| = \sqrt{10^2 + 5^2} = 5\sqrt{5}$

$$\therefore \vec{n} = \pm \frac{1}{5\sqrt{5}}(10\hat{\jmath} + 5\hat{k}) = \pm \frac{1}{\sqrt{5}}(2\hat{\jmath} + \hat{k})$$

30. ***Find $\vec{\boldsymbol{\alpha}}.(\vec{\boldsymbol{\beta}} \times \vec{\boldsymbol{\gamma}})$ if $\vec{\boldsymbol{\alpha}} = (-2, -2, 4)$, $\vec{\boldsymbol{\beta}} = (-2, 4, -2)$, and $\vec{\boldsymbol{\gamma}} = (4, -2, -2)$. Explain the geometric significance of the result.*** *[VU 2018]*

Answer:

$$\vec{\alpha}.(\vec{\beta} \times \vec{\gamma}) = \begin{vmatrix} -2 & -2 & 4 \\ -2 & 4 & -2 \\ 4 & -2 & -2 \end{vmatrix} = 0$$

This means that the three vectors are coplanar.

• Vector Calculus

Contents

2. Vector Differentiation

a. Introduction

Let us recollect the first principle of differentiation of a real-valued function $y = f(x)$ of only one variable x. The rate of change of y with respect to (w.r.t.) x is defined as:

$$y' = f'(x) = \frac{dy}{dx} = \frac{d}{dx}(f(x))$$
$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Here, we will extend this definition to vector-valued functions of single and multiple variables.

b. The Ordinary Derivative of a Vector

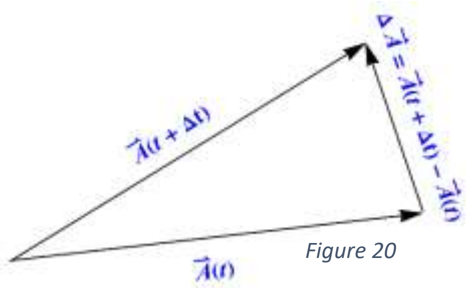


Figure 20

A vector $\vec{A}$ is a *function* of a scalar variable t (let's say time) in a specific interval or region, when there exists a value of $\vec{A}(t)$ for every value of t. If with a change from t to $t + \Delta t$, the vector changes by $\Delta\vec{A}$, then the rate of change of $\vec{A}$ with t is:

$$\frac{\Delta\vec{A}}{\Delta t} = \frac{\vec{A}(t+\Delta t) - \vec{A}(t)}{\Delta t}$$

For an infinitesimal increment of $\Delta t \to 0$, the limiting value of $\frac{\Delta\vec{A}}{\Delta t}$ w.r.t. t is called the *ordinary derivative* of the vector (the derivative exists when the limit exists):

$$\frac{d\vec{A}}{dt} = \lim_{\Delta t \to 0} \frac{\Delta\vec{A}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\vec{A}(t+\Delta t) - \vec{A}(t)}{\Delta t}$$

Since $\frac{d\vec{A}}{dt}$ itself is a vector and a function of t, we can define a derivative of $\frac{d\vec{A}}{dt}$ w.r.t t. This will be the second derivative:

$$\frac{d^2\vec{A}}{dt} = \frac{d}{dt}\left(\frac{d\vec{A}}{dt}\right)$$

Thus higher-order derivatives are defined.

Cartesian Coordinates: If a vector is expressed in terms of its Cartesian components, $\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$, then

$$\frac{d\vec{A}}{dt} = \frac{dA_x}{dt}\hat{\imath} + \frac{dA_y}{dt}\hat{\jmath} + \frac{dA_z}{dt}\hat{k}$$

C. Some Important Properties of Ordinary Derivatives of a Vector:

7. **Sum/Difference:** *If $\vec{A}$ and $\vec{B}$ are both differentiable vectors, then* $\frac{d}{dt}(\vec{A} \pm \vec{B}) = \frac{d\vec{A}}{dt} \pm \frac{d\vec{B}}{dt}$.

   Proof:

$$\frac{d}{dt}(\vec{A} \pm \vec{B}) = \lim_{\Delta t \to 0} \frac{\{\vec{A}(t+\Delta t) \pm \vec{B}(t+\Delta t)\} - \{\vec{A}(t) \pm \vec{B}(t)\}}{\Delta t}$$
$$= \lim_{\Delta t \to 0} \frac{\vec{A}(t+\Delta t) - \vec{A}(t)}{\Delta t} \pm \lim_{\Delta t \to 0} \frac{\vec{B}(t+\Delta t) - \vec{B}(t)}{\Delta t} = \frac{d\vec{A}}{dt} \pm \frac{d\vec{B}}{dt}$$

8. **Scalar Product:** *If $\vec{A}$ and $\vec{B}$ are both differentiable vectors, then* $\frac{d}{dt}(\vec{A}.\vec{B}) = \vec{A}.\frac{d\vec{B}}{dt} + \frac{d\vec{A}}{dt}.\vec{B}$

   Proof:

$$\vec{A}.\frac{d\vec{B}}{dt}+\frac{d\vec{A}}{dt}.\vec{B}=\left(A_x\hat{\imath}+A_y\hat{\jmath}+A_z\hat{k}\right).\left(\frac{dB_x}{dt}\hat{\imath}+\frac{dB_y}{dt}\hat{\jmath}+\frac{dB_z}{dt}\hat{k}\right)$$
$$+\left(\frac{dA_x}{dt}\hat{\imath}+\frac{dA_y}{dt}\hat{\jmath}+\frac{dA_z}{dt}\hat{k}\right).\left(B_x\hat{\imath}+B_y\hat{\jmath}+B_z\hat{k}\right)$$
$$=\left(A_x\frac{dB_x}{dt}+A_y\frac{dB_y}{dt}+A_z\frac{dB_z}{dt}\right)+\left(\frac{dA_x}{dt}B_x+\frac{dA_y}{dt}B_y+\frac{dA_z}{dt}B_z\right)$$
$$=\left(A_x\frac{dB_x}{dt}+\frac{dA_x}{dt}B_x\right)+\left(A_y\frac{dB_y}{dt}+\frac{dA_y}{dt}B_y\right)+\left(A_z\frac{dB_z}{dt}+\frac{dA_z}{dt}B_z\right)$$
$$=\frac{d}{dt}(A_xB_x)+\frac{d}{dt}(A_yB_y)+\frac{d}{dt}(A_zB_z)=\frac{d}{dt}(A_xB_x+A_yB_y+A_zB_z)=\frac{d}{dt}(\vec{A}.\vec{B})$$

9. **Cross Product:** *If $\vec{A}$ and $\vec{B}$ are both differentiable vectors, then* $\frac{d}{dt}(\vec{A}\times\vec{B})=\frac{d\vec{A}}{dt}\times\vec{B}+\vec{A}\times\frac{d\vec{B}}{dt}$

   <u>*Proof*</u>: Say, $\vec{A}=A_x\hat{\imath}+A_y\hat{\jmath}+A_z\hat{k}$ and $\vec{B}=B_x\hat{\imath}+B_y\hat{\jmath}+B_z\hat{k}$. Then

$$\vec{A}\times\vec{B}=\begin{vmatrix}\hat{\imath}&\hat{\jmath}&\hat{k}\\A_x&A_y&A_z\\B_x&B_y&B_z\end{vmatrix}$$

$$\Rightarrow\frac{d}{dt}(\vec{A}\times\vec{B})=\frac{d}{dt}\begin{vmatrix}\hat{\imath}&\hat{\jmath}&\hat{k}\\A_x&A_y&A_z\\B_x&B_y&B_z\end{vmatrix}=\begin{vmatrix}\hat{\imath}&\hat{\jmath}&\hat{k}\\\frac{dA_x}{dt}&\frac{dA_y}{dt}&\frac{dA_z}{dt}\\B_x&B_y&B_z\end{vmatrix}+\begin{vmatrix}\hat{\imath}&\hat{\jmath}&\hat{k}\\A_x&A_y&A_z\\\frac{dB_x}{dt}&\frac{dB_y}{dt}&\frac{dB_z}{dt}\end{vmatrix}$$
$$=\frac{d\vec{A}}{dt}\times\vec{B}+\vec{A}\times\frac{d\vec{B}}{dt}$$

10. **Product of a scalar and a vector:** *If $\vec{A}$ and ϕ are respectively a scalar and a vector and both are differentiable, then* $\frac{d}{dt}(\phi\vec{A})=\phi\frac{d\vec{A}}{dt}+\frac{d\phi}{dt}\vec{A}$

    <u>*Proof*</u>: Say $\vec{A}=A_x\hat{\imath}+A_y\hat{\jmath}+A_z\hat{k}$, then $\left[\because\frac{d\hat{\imath}}{dt}=\frac{d\hat{\jmath}}{dt}=\frac{d\hat{k}}{dt}=0\right]$

$$\frac{d}{dt}(\phi\vec{A})=\frac{d}{dt}(\phi A_x\hat{\imath}+\phi A_y\hat{\jmath}+\phi A_z\hat{k})=\frac{d}{dt}(\phi A_x)\hat{\imath}+\frac{d}{dt}(\phi A_y)\hat{\jmath}+\frac{d}{dt}(\phi A_z)\hat{k}$$
$$=\hat{\imath}\left(\frac{d\phi}{dt}A_x+\phi\frac{dA_x}{dt}\right)+\hat{\jmath}\left(\frac{d\phi}{dt}A_y+\phi\frac{dA_y}{dt}\right)+\hat{k}\left(\frac{d\phi}{dt}A_z+\phi\frac{dA_z}{dt}\right)$$
$$=\phi\frac{d\vec{A}}{dt}+\frac{d\phi}{dt}\vec{A}$$

    <u>Corollary</u>: If $\phi=$ constant, $\Rightarrow {}^{d}/_{dt}(\phi)=0\Rightarrow$

$$\frac{d}{dt}(\phi\vec{A})=\phi\frac{d\vec{A}}{dt}$$

11. **Scalar Triple Product:**

    <u>*Derivation*</u>: Say, $\vec{A}=A_x\hat{\imath}+A_y\hat{\jmath}+A_z\hat{k}$, $\vec{B}=B_x\hat{\imath}+B_y\hat{\jmath}+B_z\hat{k}$ and $\vec{C}=C_x\hat{\imath}+C_y\hat{\jmath}+C_z\hat{k}$. Then

$$\vec{A}.(\vec{B}\times\vec{C})=\begin{vmatrix}A_x&A_y&A_z\\B_x&B_y&B_z\\C_x&C_y&C_z\end{vmatrix}$$

$$\Rightarrow \frac{d}{dt}[\vec{A}.(\vec{B}\times\vec{C})] = \frac{d}{dt}\begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

$$= \begin{vmatrix} \frac{dA_x}{dt} & \frac{dA_y}{dt} & \frac{dA_z}{dt} \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} + \begin{vmatrix} A_x & A_y & A_z \\ \frac{dB_x}{dt} & \frac{dB_y}{dt} & \frac{dB_z}{dt} \\ C_x & C_y & C_z \end{vmatrix} + \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ \frac{dC_x}{dt} & \frac{dC_y}{dt} & \frac{dC_z}{dt} \end{vmatrix}$$

$$= \frac{d\vec{A}}{dt}.(\vec{B}\times\vec{C}) + \vec{A}.\left(\frac{d\vec{B}}{dt}\times\vec{C}\right) + \vec{A}.\left(\vec{B}\times\frac{d\vec{C}}{dt}\right)$$

12. Vector Triple Product:

$$\frac{d}{dt}[\vec{A}\times(\vec{B}\times\vec{C})] = \frac{d\vec{A}}{dt}\times(\vec{B}\times\vec{C}) + \vec{A}\times\left(\frac{d\vec{B}}{dt}\times\vec{C}\right) + \vec{A}\times\left(\vec{B}\times\frac{d\vec{C}}{dt}\right)$$

D. Partial Derivatives of Vectors:

Suppose $\vec{A}$ is a vector depending on more than one variable, say x, y, z for example. Then we write $\vec{A} = \vec{A}(x, y, z)$. The partial derivative of $\vec{A}$ w.r.t. x is defined and denoted as follows when the limit exists:

$$\frac{\partial\vec{A}}{\partial x} = \lim_{\Delta x\to 0}\frac{\vec{A}(x+\Delta x, y, z) - \vec{A}(x, y, z)}{\Delta x}$$

Similarly, the following are the partial derivatives of $\vec{A}$ w.r.t. y and z, respectively, when the limits exist:

$$\frac{\partial\vec{A}}{\partial y} = \lim_{\Delta y\to 0}\frac{\vec{A}(x, y+\Delta y, z) - \vec{A}(x, y, z)}{\Delta y}$$

$$\frac{\partial\vec{A}}{\partial z} = \lim_{\Delta z\to 0}\frac{\vec{A}(x, y, z+\Delta z) - \vec{A}(x, y, z)}{\Delta z}$$

The remarks on continuity and differentiability of functions of one variable can be extended to two or more variables. So, if $\vec{A} = \vec{A}(x, y, z)$, and the variables change infinitesimally to $x + dx, y + dy$, and $z + dz$, then the total infinitesimal change in $\vec{A}$:

$$d\vec{A} = \frac{\partial\vec{A}}{\partial x}dx + \frac{\partial\vec{A}}{\partial y}dy + \frac{\partial\vec{A}}{\partial z}dz$$

C. Scalar and Vector Fields

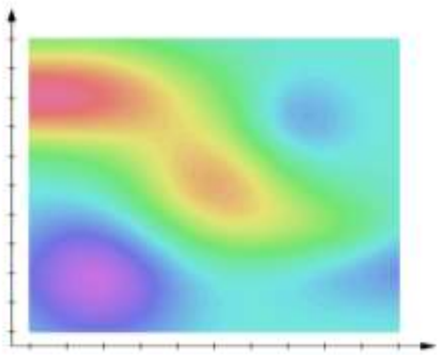


Figure 21

By Lucas Vieira - Own work, Public Domain, https://commons.wikimedia.org/w/index.php?curid=20462138

When a quantity changes from one point to another in a region of space, it can be expressed as a function of position in that region, and that region is then called the *field of that quantity.* These are of two types:

- ***Scalar Fields***: When the value of a scalar quantity ϕ (it may be a dimensionless mathematical number or a physical quantity) changes continuously from one point to another, e.g. with x, y, and z in the physical space, then $\phi = \phi(x, y, z)$

is the field of that scalar.

In Figure 2 there is a scalar field such as *temperature, electric potential,* or *pressure*, where the intensity of the field is represented by different hues of colors. Clearly, as the field is uniquely determined by the magnitude of the scalar at each position, it is independent of a coordinate system.

- ***Vector Fields***: In this case, if the continuously changing quantity from point to point is a vector $\vec{A} = \vec{A}(x, y, z)$, then every point the corresponding region has a specific vector (instead of a number) associated with it. In terms of coordinates, a vector field in n-dimensional space can be represented as a function that associates an n-tuple of real numbers to each point of the domain (e.g. A_x, A_y, A_z for three dimensions). This representation of a vector field depends on the coordinate system, and there is a well-defined transformation law in passing from one coordinate system to the other.

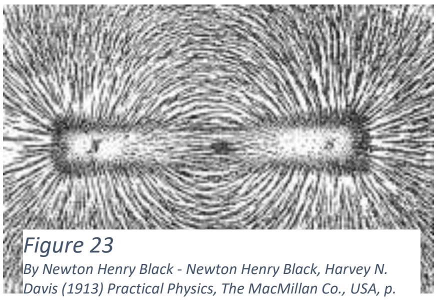


Figure 23
By Newton Henry Black - Newton Henry Black, Harvey N. Davis (1913) Practical Physics, The MacMillan Co., USA, p. 242, fig. 200, Public Domain, https://commons.wikimedia.org/w/index.php?curid=73846

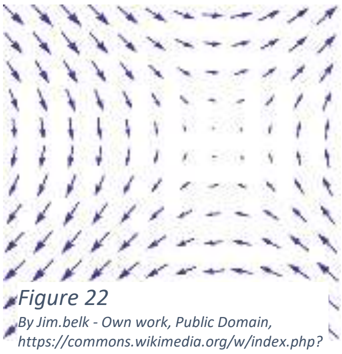


Figure 22
By Jim.belk - Own work, Public Domain, https://commons.wikimedia.org/w/index.php?curid=8008790

In the adjacent figure (Figure 3), we see a two-dimensional vector field $\vec{A} = \sin y\,\hat{\imath} + \sin x\,\hat{\jmath}$. Examples of a vector field are electric field, magnetic field, gravitational field, etc. In Figure 4 we see the scattered iron fillings rearranging themselves around a magnet, depicting the magnetic field. We call them magnetic field lines.

d. Vector Differential Operator

In mathematics, an operator is generally a mapping or function that acts on elements of a space to produce elements of the same or other space. The vector differential operator

$$\vec{\nabla} = \frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k} = \hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

is called *Del* or *Nabla*. For most cases, this operator has properties similar to that of ordinary vectors. In its current form, it has no meaning, as it has not been applied to anything. To understand its effect and meaning, we need to 'operate' it on a scalar or a vector.

A. Gradient

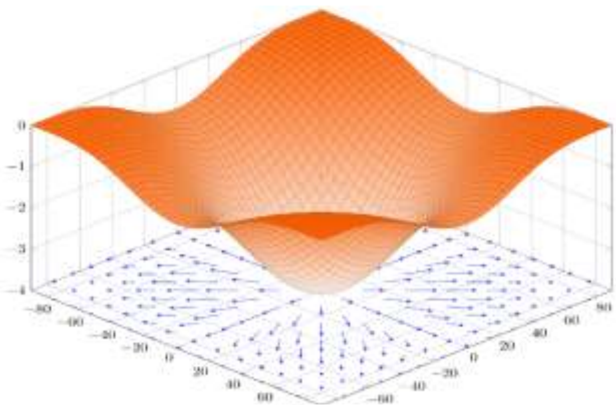


Figure 24
By MartinThoma - Own work, CC0, https://commons.wikimedia.org/w/index.php?curid=71375503

Let $\phi(x, y, z)$ be a scalar function defined and differentiable at each point (x, y, z) in a certain region of space (i.e. ϕdefines a differentiable scalar field). Then the <u>*gradient*</u> of ϕ, written '$\nabla\phi$' or '$grad\ \phi$', is defined as

$$\vec{\nabla}\phi = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\phi$$
$$= \hat{\imath}\frac{\partial\phi}{\partial x} + \hat{\jmath}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

In Figure 5, the gradient of the function $f(x, y) = -(\cos^2 x + \cos^2 y)^2$ is depicted as a projected vector field on the bottom plane.

Note: *del*, operating on a scalar field, creates a vector field. The direction of the gradient of a function denotes the direction in which *the function changes maximally* and the magnitude of the gradient shows the rate of change of the function w.r.t position in that direction.

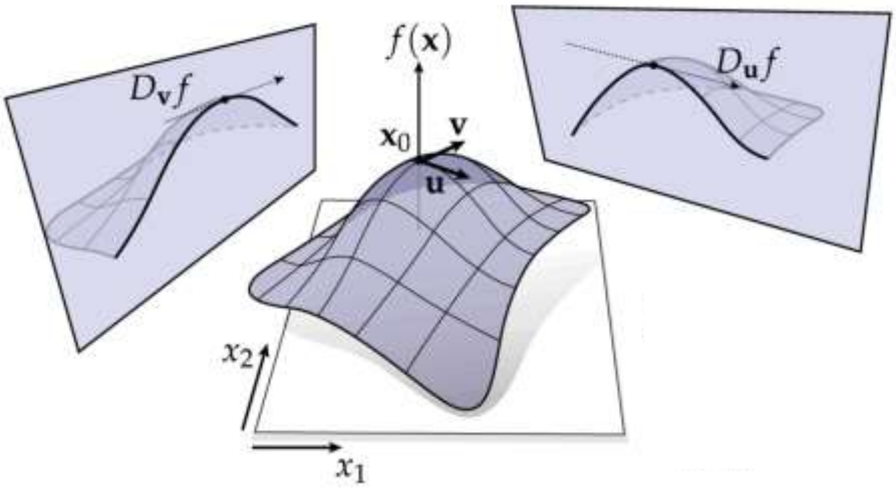


Figure 25
http://15462.courses.cs.cmu.edu/fall2018/lecture/vectorcalc/slide_024

Directional Derivatives:

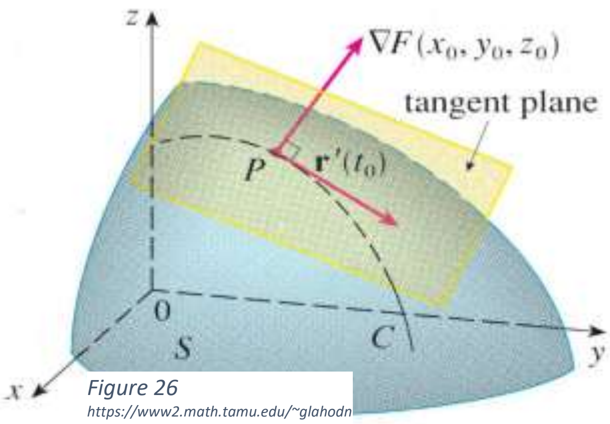


Figure 26
https://www2.math.tamu.edu/~glahodny/Math251/Section%2012.6.pdf

How do we think about derivatives of a function which has multiple variables? Let's say we have a function $f(x_1, x_2)$. We can cut a slice through the function along some line, i.e. some arbitrary direction. Figure 25 shows this function which is cut twice, along the direction of two different vectors $\vec{u}$ and $\vec{v}$, respectively. The directional derivative of f at a point with position vector $\overrightarrow{x_0}$ in the direction of $\vec{u}$ is

$$D_{\vec{u}}f(\overrightarrow{x_0}) = \lim_{\varepsilon\to 0}\frac{f(\overrightarrow{x_0} + \varepsilon\,\vec{u}) - f(\overrightarrow{x_0})}{\varepsilon} = \lim_{\varepsilon\to 0}\frac{f(x_0^1 + \varepsilon\,u^1,\ x_0^2 + \varepsilon\,u^2) - f(x_0^1,\ x_0^2)}{\varepsilon}$$

where ε is a small increment in the direction of $\vec{u}$. Similarly, we can define $D_{\vec{v}}f$ as well.

Given this definition of directional derivatives, the directional derivative of a scalar function $\phi(x, y, z)$ in the direction of a vector $\vec{A}$ is $D_{\vec{A}}\phi$. If the unit vector in the direction of $\vec{A}$ is $\hat{a} = \vec{A}/|\vec{A}|$, then

$$D_{\vec{A}}\phi = \vec{\nabla}\phi.\hat{a}$$

In other words, the directional derivative of a scalar function $\phi(x, y, z)$ at a point (x_1, y_1, z_1) in the direction of a unit vector $\hat{n}$ is $D_{\hat{n}}\phi = \left\{\left(\vec{\nabla}\phi\right)_{(x_1, y_1, z_1)}\right\}.\hat{n}$

B. Divergence

Suppose $\vec{A}(x,y,z) = A_x(x,y,z)\hat{\imath} + A_y(x,y,z)\hat{\jmath} + A_z(x,y,z)\hat{k}$ is defined and differentiable at each point (x,y,z) in a region of space (i.e., $\vec{A}$ defines a differentiable vector field). Then the ***divergence of*** $\vec{A}$ is defined as

$$\vec{\nabla}.\vec{A} = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\left(A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}\right) = \frac{\partial A_x}{\partial z} + \frac{\partial A_y}{\partial z} + \frac{\partial A_z}{\partial z}$$

Hence, the divergence of a vector is a scalar function.

Note: $\vec{\nabla}.\vec{A} \neq \vec{A}.\vec{\nabla}$. So, while $\vec{A}$ is a vector, $\vec{\nabla}.\vec{A}$ is a scalar but $\vec{A}.\vec{\nabla}$ is an operator.

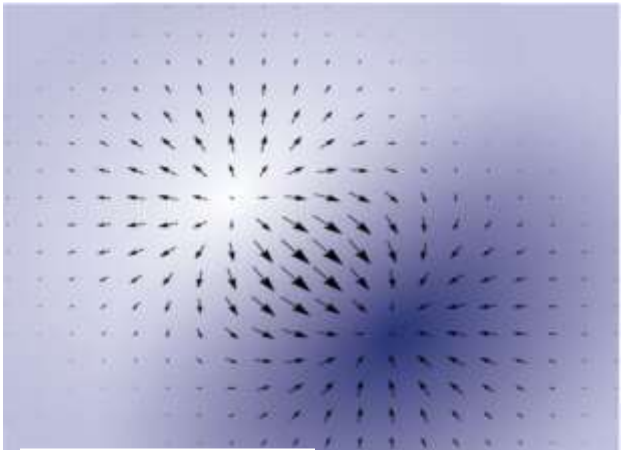


Figure 27
http://15462.courses.cs.cmu.edu/fall2018/lecture/vectorcalc/slide_039

Flux:

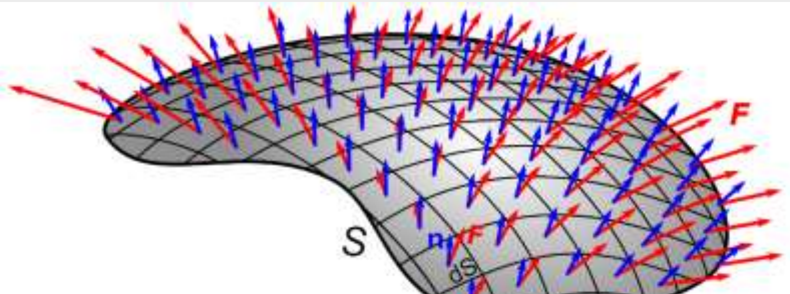


Figure 28
By Chetvorno - Own work, CC0, https://commons.wikimedia.org/w/index.php?curid=82743721

In vector calculus, the *flux of a vector field* is a scalar quantity, defined as the surface integral of the perpendicular component of a vector field over a surface. To calculate the flux of a vector field $\boldsymbol{F}$ (red arrows) through a surface $\boldsymbol{S}$, the surface is divided into small patches $d\boldsymbol{S}$. The flux through each patch is equal to the normal (perpendicular) component of the field, the dot product of $\boldsymbol{F}(\vec{x})$ with the unit normal vector $\boldsymbol{n}(\vec{x})$ (blue arrows) at the point $\vec{x}$ multiplied by the area $d\boldsymbol{S}$. The sum of $\boldsymbol{F}.\boldsymbol{n}d\boldsymbol{S}$ for each patch on the surface is the flux through the surface.

Divergence of a vector field signifies the outward flux per unit volume. In Figure 8 we have superposed pictures of two fields. The first one is a vector field $\vec{A}$ represented by arrows of varying lengths. The other is a scalar field representing $\vec{\nabla}.\vec{A}$, with positive to negative values of it represented by a shade from white to blue. Clearly, there is a large outward flux at one point (white region) and a large incoming flux at another (blue region). The first one is called a *source* and the second one is called a *sink* of the vector field. When $\vec{\nabla}.\vec{A} = 0$, the field is called *solenoidal*.

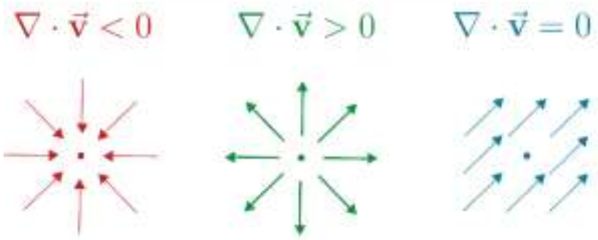


Figure 29
https://www.khanacademy.org/math/multivariable-calculus/multivariable-derivatives/divergence-and-curl-articles/a/divergence

C. Curl

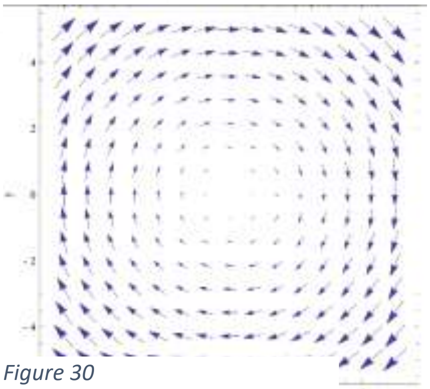


Figure 30
By Loodog at English Wikipedia, CC BY 2.5, https://commons.wikimedia.org/w/index.php?curid=2438212

Suppose $\vec{A}(x,y,z) = A_x(x,y,z)\hat{\imath} + A_y(x,y,z)\hat{\jmath} + A_z(x,y,z)\hat{k}$ defines a differentiable vector field. Then the ***Curl of $\vec{A}$*** is defined as:

$$\vec{\nabla}\times\vec{A} = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\times\left(A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}\right)$$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$= \hat{\imath}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{\jmath}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{k}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

Note: In the expansion of the determinant the operators $\frac{\partial}{\partial x_i}$ must precede A_i.

The curl of a vector field at a specific point is also a vector, whose length and direction denote the magnitude and axis of the maximum circulation of the field calculated at that point. The curl of a field is formally defined as the circulation density at each point of the field. Figure 11 shows a 2-dimensional vector with a uniform curl. A vector with zero curl ($\vec{\nabla}\times\vec{A} = 0$) is called an *Irrotational vector*.

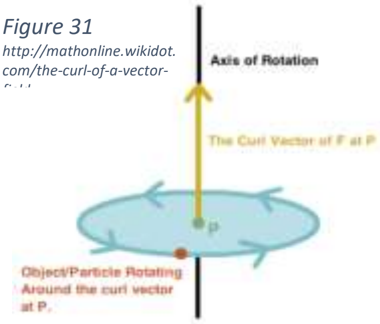


Figure 31
http://mathonline.wikidot.com/the-curl-of-a-vector-...

D. Some Important Properties

We have already seen that gradient of a scalar ϕ and curl of a vector $\vec{A}$ are both vectors. Hence we can calculate the divergence and curl of both of these. Similarly, as the divergence of a vector is a scalar, we can calculate the gradient of that.

- $Div.Grad.\phi = \vec{\nabla}.(\vec{\nabla}\phi)$

$$= \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\left(\hat{\imath}\frac{\partial\phi}{\partial x} + \hat{\jmath}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right) = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi = \nabla^2\phi$$

Here, $\nabla^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)$ is called the *Laplacian* /*Laplacian Operator*, which is a scalar operator.

- $Curl.Grad.\phi = \vec{\nabla}\times(\vec{\nabla}\phi)$

$$= \vec{\nabla}\times\left(\hat{\imath}\frac{\partial\phi}{\partial x} + \hat{\jmath}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix}$$

$$= \hat{\imath}\left\{\frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial y}\right)\right\} + \hat{\jmath}\left\{\frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial x}\right) - \frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial z}\right)\right\} + \hat{k}\left\{\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial x}\right)\right\}$$

$$= \hat{\imath}\left(\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial z\partial y}\right) + \hat{\jmath}\left(\frac{\partial^2\phi}{\partial z\partial x} - \frac{\partial^2\phi}{\partial x\partial z}\right) + \hat{k}\left(\frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial y\partial x}\right) = \mathbf{0}$$

- $Grad.Div.\vec{A} = \vec{\boldsymbol{\nabla}}(\vec{\boldsymbol{\nabla}}.\vec{\boldsymbol{A}})$

$$= \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(\frac{\partial A_x}{\partial z} + \frac{\partial A_y}{\partial z} + \frac{\partial A_z}{\partial z}\right) = \boldsymbol{Do\ it\ at\ home}$$

- $Div.Curl.\vec{A} = \vec{\boldsymbol{\nabla}}.(\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}})$

$$= \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$= \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\left\{\hat{\imath}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{\jmath}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{k}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\right\}$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$= \frac{\partial^2 A_z}{\partial x\partial y} - \frac{\partial^2 A_y}{\partial x\partial z} + \frac{\partial^2 A_x}{\partial y\partial z} - \frac{\partial^2 A_z}{\partial y\partial x} + \frac{\partial^2 A_y}{\partial z\partial x} - \frac{\partial^2 A_x}{\partial z\partial y} = \mathbf{0}$$

- $Curl.Curl.\vec{A} = \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla}.\vec{A}) - \nabla^2\vec{A}$

  This expression does not help us much, other than defining a Vector Laplacian $\nabla^2\vec{A} = (\nabla^2 A_x, \nabla^2 A_y, \nabla^2 A_z)$.

$$\therefore\ \boldsymbol{\nabla^2\vec{A}} = \vec{\boldsymbol{\nabla}}(\vec{\boldsymbol{\nabla}}.\vec{\boldsymbol{A}}) - \vec{\boldsymbol{\nabla}} \times (\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}})$$

E. Other Important Properties

- $\vec{\boldsymbol{\nabla}}.(\boldsymbol{\phi}\vec{\boldsymbol{A}})$

$$= \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).(\phi A_x\hat{\imath} + \phi A_y\hat{\jmath} + \phi A_z\hat{k}) = \frac{\partial}{\partial x}(\phi A_x) + \frac{\partial}{\partial y}(\phi A_y) + \frac{\partial}{\partial z}(\phi A_z)$$

$$= \phi\left(\frac{\partial A_x}{\partial z} + \frac{\partial A_y}{\partial z} + \frac{\partial A_z}{\partial z}\right) + \left(A_x\frac{\partial \phi}{\partial x} + A_y\frac{\partial \phi}{\partial y} + A_z\frac{\partial \phi}{\partial z}\right)$$

$$= \phi(\vec{\nabla}.\vec{A}) + (A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}).\left(\hat{\imath}\frac{\partial \phi}{\partial x} + \hat{\jmath}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}\right) = \boldsymbol{\phi}(\vec{\boldsymbol{\nabla}}.\vec{\boldsymbol{A}}) + \vec{\boldsymbol{A}}.\vec{\boldsymbol{\nabla}}\boldsymbol{\phi}$$

- $\vec{\boldsymbol{\nabla}} \times (\boldsymbol{\phi}\vec{\boldsymbol{A}}) = \vec{\boldsymbol{\nabla}}\boldsymbol{\phi} \times \vec{\boldsymbol{A}} + \boldsymbol{\phi}\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}}$ (Prove this)
- $\vec{\boldsymbol{\nabla}}.(\vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}}) = \vec{\boldsymbol{A}}.(\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{B}}) + \vec{\boldsymbol{B}}.(\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}})$ (Prove this)

e. Solved Questions

31. **If $\vec{A} = x\hat{\imath} + y^2\hat{\jmath} + z^3\hat{k}$. Find $d\vec{A}$**

Answer: $\vec{A} = x\hat{\imath} + y^2\hat{\jmath} + z^3\hat{k}$.

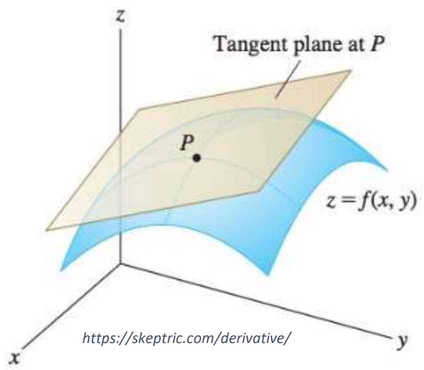
$\therefore \frac{\partial \vec{A}}{\partial x} = \hat{\imath}, \frac{\partial \vec{A}}{\partial y} = 2y\hat{\jmath}$, and $\frac{\partial \vec{A}}{\partial z} = 3z^2\hat{k}$

Hence, the change in $\vec{A}$ is:

$$d\vec{A} = \frac{\partial \vec{A}}{\partial x}dx + \frac{\partial \vec{A}}{\partial y}dy + \frac{\partial \vec{A}}{\partial z}dz = \hat{\imath}\ dx + \hat{\jmath}\ 2ydy + \hat{k}\ 3z^2dz$$

32. ***Show that $\vec{\nabla}\phi$ is a perpendicular vector on the plane denoted by $\phi(x,y,z) = c$ (c is a constatnt)***

Answer:



Let's say P is a point on the plane $\phi(x,y,z) = c$, with coordinates (x,y,z). Hence, the position vector of P is $\vec{r} = x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}$. So, the vector $d\vec{r} = dx\,\hat{\imath} + dy\,\hat{\jmath} + dz\,\hat{k}$ denoting the infinitesimal change in $\vec{r}$, lies in the tangent plane to the plane ϕ at P.

Now,

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$$
$$= \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\left(dx\,\hat{\imath} + dy\,\hat{\jmath} + dz\,\hat{k}\right)$$
$$= \left(\vec{\nabla}\phi\right).d\vec{r} = 0$$
$$[\because\ \phi(x,y,z) = c \Rightarrow d\phi = 0]$$

i.e. $\left(\vec{\nabla}\phi\right)$ is perpendicular to $d\vec{r}$ (for any $d\vec{r}$). $d\vec{r}$ lies in the tangent plane and thus $\vec{\nabla}\phi$ must be perpendicular to the tangent plane itself $\Rightarrow$ the gradient is perpendicular to the surface $\phi(x,y,z) = c$.

33. ***Expand and simplify the form of $\vec{\nabla}(\vec{\nabla}.\vec{A})$…***

34. ***If $\phi(x,y,z) = xy^2z^3$, then find $\vec{\nabla}\phi$ at the point $(4,-1,1)$***

Answer:

$\phi(x,y,z) = xy^2z^3$, hence $\frac{\partial\phi}{\partial x} = y^2z^3$, $\frac{\partial\phi}{\partial y} = 2xyz^3$, and $\frac{\partial\phi}{\partial z} = 3xy^2z^2$

$$\therefore \vec{\nabla}\phi = \hat{\imath}\frac{\partial\phi}{\partial x} + \hat{\jmath}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z} = y^2z^3\hat{\imath} + 2xyz^3\hat{\jmath} + 3xy^2z^2\hat{k}$$
$$\therefore \left(\vec{\nabla}\phi\right)_{(4,-1,1)} = \hat{\imath} - 8\hat{\jmath} + 12\hat{k}$$

35. ***If $\phi(x,y,z) = x^2y^2z^2$, then find the directional derivative of ϕ at the point $(1,-1,2)$ in the direction of the vector $(\hat{\imath} - 2\hat{\jmath} + 2\hat{k})$.***

Answer:

$\phi(x,y,z) = x^2y^2z^2$, hence $\frac{\partial\phi}{\partial x} = 2xy^2z^2$, $\frac{\partial\phi}{\partial y} = 2x^2yz^2$, and $\frac{\partial\phi}{\partial z} = 2x^2y^2z$

$$\therefore \vec{\nabla}\phi = \hat{\imath}\frac{\partial\phi}{\partial x} + \hat{\jmath}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z} = 2xy^2z^2\hat{\imath} + 2x^2yz^2\hat{\jmath} + 2x^2y^2z\hat{k}$$
$$\therefore \left(\vec{\nabla}\phi\right)_{(1,-1,2)} = 8\hat{\imath} - 8\hat{\jmath} + 4\hat{k}$$

Now the unit vector in the direction of $(\hat{\imath} - 2\hat{\jmath} + 2\hat{k})$

$$\hat{n} = \frac{\hat{\imath} - 2\hat{\jmath} + 2\hat{k}}{\sqrt{1+4+4}} = \frac{\hat{\imath} - 2\hat{\jmath} + 2\hat{k}}{3}$$

Hence, the directional derivative of $\phi(x,y,z)$ at the point $(1,-1,2)$ in the direction of $\hat{n}$

$$\left\{\left(\vec{\nabla}\phi\right)_{(1,-1,2)}\right\}.\hat{n} = (8\hat{\imath} - 8\hat{\jmath} + 4\hat{k}).\left(\frac{\hat{\imath} - 2\hat{\jmath} + 2\hat{k}}{3}\right)$$
$$= \frac{8+16+8}{3} = \frac{32}{3}$$

36. ***If $\vec{r} = x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}$, find $\vec{\nabla}\left(\frac{1}{r}\right)$.*** [CU 2014]

Answer:

Method 1:

$\vec{r} = x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}, \therefore r = \sqrt{x^2+y^2+z^2} \Rightarrow \frac{1}{r} = (x^2+y^2+z^2)^{-\frac{1}{2}}$

$$\therefore \vec{\nabla}\left(\frac{1}{r}\right) = \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right)\left(\frac{1}{r}\right) = \frac{\partial}{\partial x}\left(\frac{1}{r}\right)\hat{\imath} + \frac{\partial}{\partial y}\left(\frac{1}{r}\right)\hat{\jmath} + \frac{\partial}{\partial z}\left(\frac{1}{r}\right)\hat{k}$$

Now,

$$\frac{\partial}{\partial x}\left(\frac{1}{r}\right) = \frac{\partial}{\partial r}\left(\frac{1}{r}\right)\frac{\partial r}{\partial x} = -\frac{1}{r^2}\frac{\partial}{\partial x}\left(\sqrt{x^2+y^2+z^2}\right) = \left(-\frac{1}{r^2}\right)\frac{1}{2}\,(x^2+y^2+z^2)^{-\frac{1}{2}}\,2x$$

$$= -\frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}} = -\frac{x}{r^3}$$

Similarly, $\frac{\partial}{\partial y}\left(\frac{1}{r}\right) = -\frac{y}{r^3}$ and $\frac{\partial}{\partial z}\left(\frac{1}{r}\right) = -\frac{z}{r^3}$

$$\therefore\ \vec{\boldsymbol{\nabla}}\left(\frac{\boldsymbol{1}}{\boldsymbol{r}}\right) = \left(-\frac{x}{r^3}\right)\hat{\imath} + \left(-\frac{y}{r^3}\right)\hat{\jmath} + \left(-\frac{z}{r^3}\right)\hat{k} = -\frac{x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}}{r^3} = -\frac{\vec{\boldsymbol{r}}}{\boldsymbol{r}^3}$$

Method 2:

We know $\vec{\boldsymbol{\nabla}}\boldsymbol{r}^{\boldsymbol{n}} = \boldsymbol{n}\,\boldsymbol{r}^{\boldsymbol{n-2}}\,\vec{\boldsymbol{r}}$ (see the next problem)

here $n = -1$.

$$\therefore \vec{\nabla}\left(\frac{1}{r}\right) = (-1)(r^{-1-2})\vec{r} = -\frac{\vec{r}}{r^3}$$

37. ***Show that*** $\vec{\boldsymbol{\nabla}}\boldsymbol{r}^{\boldsymbol{n}} = \boldsymbol{n}\,\boldsymbol{r}^{\boldsymbol{n-2}}\,\vec{\boldsymbol{r}}$.

Answer:

$$\vec{\nabla}(r^n) = \vec{\nabla}\left(\sqrt{x^2+y^2+z^2}\right)^n = \vec{\nabla}(x^2+y^2+z^2)^{\frac{n}{2}}$$

$$= \frac{\partial}{\partial x}\left((x^2+y^2+z^2)^{\frac{n}{2}}\right)\hat{\imath} + \frac{\partial}{\partial y}\left((x^2+y^2+z^2)^{\frac{n}{2}}\right)\hat{\jmath}$$

$$+ \frac{\partial}{\partial z}\left((x^2+y^2+z^2)^{\frac{n}{2}}\right)\hat{k}$$

$$= \left(\frac{n}{2}(x^2+y^2+z^2)^{\frac{n}{2}-1}\,2x\right)\hat{\imath} + \left(\frac{n}{2}(x^2+y^2+z^2)^{\frac{n}{2}-1}\,2y\right)\hat{\jmath}$$

$$+ \left(\frac{n}{2}(x^2+y^2+z^2)^{\frac{n}{2}-1}\,2z\right)\hat{k} = n(x^2+y^2+z^2)^{\frac{n-2}{2}}(x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k})$$

$$= n(r)^{n-2}\,\vec{r}$$

38. ***Find the Directional Derivative of*** $\boldsymbol{\phi} = \left(\frac{\boldsymbol{1}}{\boldsymbol{r}}\right)$ ***along*** $\vec{\boldsymbol{r}} = \boldsymbol{x}\,\hat{\boldsymbol{\imath}} + \boldsymbol{y}\,\hat{\boldsymbol{\jmath}} + \boldsymbol{z}\,\hat{\boldsymbol{k}}$.

Answer:

$\vec{\nabla}\phi = -\frac{\vec{r}}{r^3}$ (from problem 6 above)

Hence, the directional derivative of $\vec{\nabla}\phi$ in the direction of $\vec{r}$

$$= \vec{\nabla}\phi.\hat{r} = -\frac{\vec{r}}{r^3}.\hat{r} = -\frac{r}{r^3}\hat{r}.\hat{r} = -\frac{1}{r^2}$$

39. ***If*** $\vec{\boldsymbol{r}} = \boldsymbol{x}\,\hat{\boldsymbol{\imath}} + \boldsymbol{y}\,\hat{\boldsymbol{\jmath}} + \boldsymbol{z}\,\hat{\boldsymbol{k}}$***, find*** $(\vec{\boldsymbol{\nabla}}.\vec{\boldsymbol{r}})$.

Answer:

$$\vec{\nabla}.\vec{r} = \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right).(x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}) = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1+1+1 = 3$$

40. ***If*** $\vec{\boldsymbol{r}} = \boldsymbol{x}\,\hat{\boldsymbol{\imath}} + \boldsymbol{y}\,\hat{\boldsymbol{\jmath}} + \boldsymbol{z}\,\hat{\boldsymbol{k}}$***, find*** $(\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{r}})$.

Answer:

$$\vec{\nabla} \times \vec{r} = \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right) \times (x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$$= \hat{\imath}\left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right) + \hat{\jmath}\left(\frac{\partial x}{\partial z} - \frac{\partial z}{\partial x}\right) + \hat{k}\left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right) = 0$$

41. ***If $\vec{\omega}$ is a constant vector, $\vec{r}$ is the position vector, and $\vec{v} = \vec{\omega} \times \vec{r}$, then show that $\vec{\omega} = \frac{1}{2}(\vec{\nabla} \times \vec{v})$.*** [VU 2011]

Answer:

Let's say that the cartesian components of the vector $\vec{\omega}$ are respectively ω_x, ω_y, and ω_z. Then,

$$\vec{v} = \vec{\omega} \times \vec{r} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \omega_x & \omega_y & \omega_z \\ x & y & z \end{vmatrix} = \hat{\imath}(\omega_y z - \omega_z y) + \hat{\jmath}(\omega_z x - \omega_x z) + \hat{k}(\omega_x y - \omega_y x)$$

$$\therefore \frac{1}{2}(\vec{\nabla} \times \vec{v}) = \frac{1}{2}\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (\omega_y z - \omega_z y) & (\omega_z x - \omega_x z) & (\omega_x y - \omega_y x) \end{vmatrix}$$

$$= \hat{\imath}\left(\frac{\partial(\omega_x y - \omega_y x)}{\partial y} - \frac{\partial(\omega_z x - \omega_x z)}{\partial z}\right)$$

$$+ \hat{\jmath}\left(\frac{\partial(\omega_y z - \omega_z y)}{\partial z} - \frac{\partial(\omega_x y - \omega_y x)}{\partial x}\right)$$

$$+ \hat{k}\left(\frac{\partial(\omega_z x - \omega_x z)}{\partial x} - \frac{\partial(\omega_y z - \omega_z y)}{\partial y}\right)$$

$$= \frac{1}{2}\left[\hat{\imath}(\omega_x + \omega_x) + \hat{\jmath}(\omega_y + \omega_y) + \hat{k}(\omega_z + \omega_z)\right] = \vec{\omega}\ (\boldsymbol{QED})$$

42. ***If $\vec{\omega}$ is a constant vector, $\vec{r}$ is the position vector, and $\vec{v} = \vec{\omega} \times \vec{r}$, then show that $\vec{\nabla}.\vec{v} = 0$*** [VU 2012]

Answer:

$\because \vec{\nabla}.(\vec{A} \times \vec{B}) = \vec{B}.(\vec{\nabla} \times \vec{A}) - \vec{A}.(\vec{\nabla} \times \vec{B})$,

$\vec{\nabla} \times \vec{r} = 0$ [see problem 10],

and $\vec{\nabla} \times \vec{\omega} = 0$ [$\vec{\omega}$ is a constant vector],

$$\vec{\nabla}.\vec{v} = \vec{\nabla}.(\vec{\omega} \times \vec{r}) = \vec{\omega}.(\vec{\nabla} \times \vec{r}) - \vec{r}.(\vec{\nabla} \times \vec{\omega})$$

$$= 0 - 0 = 0$$

43. ***If $\vec{A} = x^2 z\,\hat{\imath} + 2y^3 z^2\,\hat{\jmath} + xy^2\,\hat{k}$, what is $\vec{\nabla}.\vec{A}$ at the point $(1, -1, 1)$?***

Answer:

$$\vec{\nabla}.\vec{A} = \frac{\partial}{\partial x}(x^2 z) + \frac{\partial}{\partial y}(2y^3 z^2) + \frac{\partial}{\partial z}(xy^2) = 2xz + 6y^2 z^2 + 0$$

$$\therefore \vec{\nabla}.\vec{A}\Big|_{(1,-1,1)} = 2 + 6 = 8$$

44. ***More problems to come…***

3. Vector Integration

a. Introduction

The reader is expected to be familiar with the integration of real-valued function $f(x)$ of one variable, especially the indefinite integral

$$\int f(x)\,dx$$

and the definite integral on a closed interval

$$\int_a^b f(x)\,dx$$

Here we extend these definitions to vector-valued functions.

b. Ordinary Integration

Let's say $\vec{A}(t) = \hat{\imath}\,A_x(t) + \hat{\jmath}\,A_y(t) + \hat{k}\,A_z(t)$ is a vector function of a scalar variable 't', where A_p is the component of $\vec{A}$ in the direction of the axis 'p'. Then the indefinite integral of $\vec{A}$ w.r.t. t is

$$\int \vec{A}(t)\,dt = \int [\hat{\imath}\,A_x(t) + \hat{\jmath}\,A_y(t) + \hat{k}\,A_z(t)]\,dt = \hat{\imath}\int A_x(t)\,dt + \hat{\jmath}\int A_y(t)\,dt + \hat{k}\int A_z(t)\,dt$$

If there exists a vector function $\vec{B}(t)$ such that $\vec{A}(t) = d\left(\vec{B}(t)\right)\big/_{dt}$, then

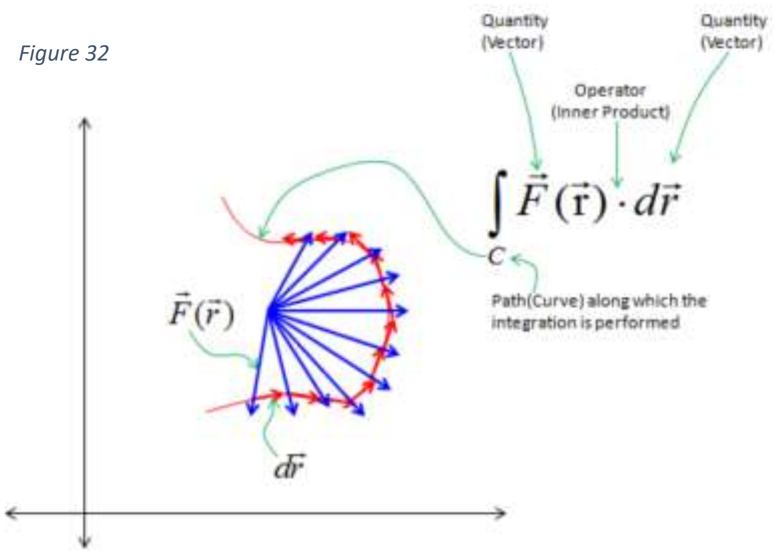
$$\int \vec{A}(t)\,dt = \vec{B}(t) + \vec{c}$$

where $\vec{c}$ is an arbitrary constant vector (independent of t), and the definite integration

$$\int_a^b \vec{A}(t)\,dt = \int_a^b d\left(\vec{B}(t)\right) = \left(\vec{B}(t) + \vec{c}\right)\Big|_a^b = \vec{B}(b) - \vec{B}(a)$$

c. Line Integral

Figure 32



Suppose $\vec{r}(t) = x(t)\,\hat{\imath} + y(t)\,\hat{\jmath} + z(t)\,\hat{k}$ is the position vector of points $P(x,y,z)$ and suppose $\vec{r}(t)$ defines a curve C joining points P_1 and P_2 where $t = t_1$ and t_2 respectively. We assume that C is composed of a finite number of curves for each of which $\vec{r}(t)$ has a continuous derivative. Let $\vec{F}(x,y,z) = F_x(x,y,z)\,\hat{\imath} + F_y(x,y,z)\,\hat{\jmath} + F_z(x,y,z)\,\hat{k}$ be a vector function of position defined and continuous along C. Then the integral of the tangential component of $\vec{F}$ along C from P_1 to P_2, written as

$$\int_C \vec{F}(\vec{r}).d\vec{r} = \int_{P_1}^{P_2} \vec{F}(\vec{r}).d\vec{r} = \int_C F_x dx + F_y dy + F_z dz$$

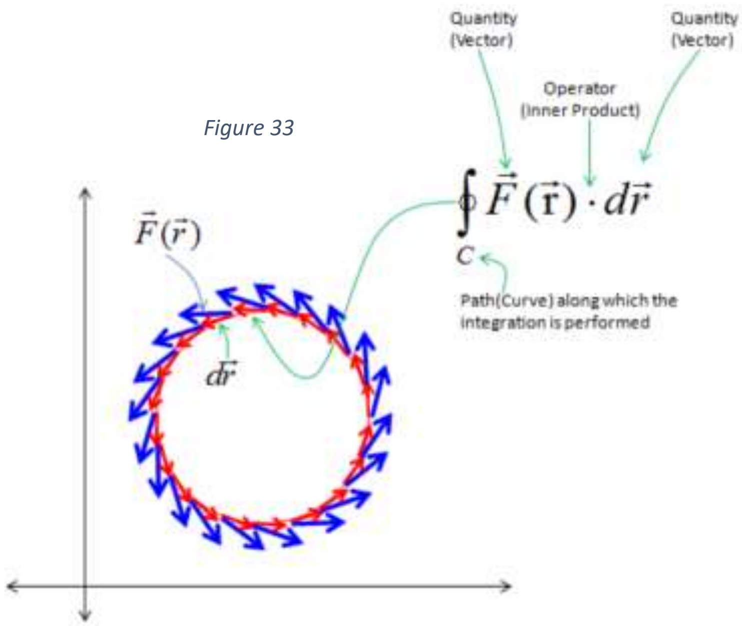
is an example of a line integral.

If C is a closed curve (we suppose a simple closed curve, i.e. it doesn't intersect itself anywhere), the integral around C is often denoted by

$$\oint \vec{F}(\vec{r}).d\vec{r} = \oint F_x dx + F_y dy + F_z dz$$

Hence, the line integral of a vector function is a scalar quantity. When $\vec{F}(\vec{r})$ is a force on a particle moving along C, this line integral represents the *work done by the force*. In Fluid dynamics and aerodynamics, where $\vec{F}$ represents the velocity of the fluid, this integral is called the *circulation of $\vec{F}$ about C*. In general, any integral that is to be integrated along a curve is called a line integral. Such integrals can be defined in terms of the Riemann sum (limiting sum) of elementary calculus.

Figure 33



Some Comments about Functions: (http://www.sharetechnote.com/html/Calculus_Integration_Line.bak)

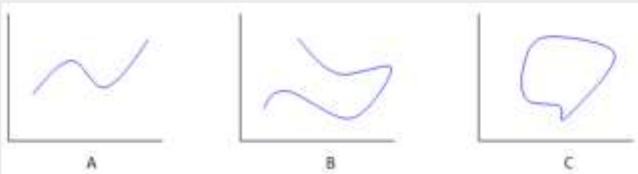
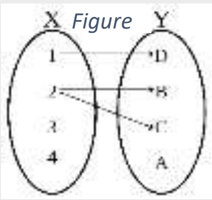
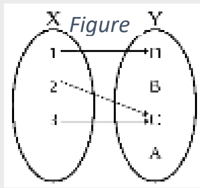


Figure 34



Figure

The concept of line integral probably doesn't sound very complicated to you, but given a specific real mathematical application using line integral, you probably don't know what to do. A possible reason for this kind of difficulty is not because of the concept of line integral itself, but because of not knowing to represent the curve (path) in a function. The pre-college definition of a function is consistent with Figure 16, where one-to-one or one-to-many valued mappings are allowed, it fails in the case of Figure 17, which is a many-to-one map (Figures from Bin im Garten - Own work,). Now check out Figure 34. Plot A definitely can be a function since one value on the x axis (independent variable) is mapped to one value on y axis (dependent variable). How about plot B? It cannot be: at least according what we have learned in most of high school math. In this example, some x value maps to two different y value. Same is true for Plot C. Here lies the problem. Most of the curves we deal with look like plot B or C. We can represent this type of curves in one of the following type of functions:



Figure

i. Parametric Function
ii. Vector Function
iii. Complex Function (Function with complex variable)

Now you may see why most of applications of line integral are given in the form of one of these.

d. Surface Integral

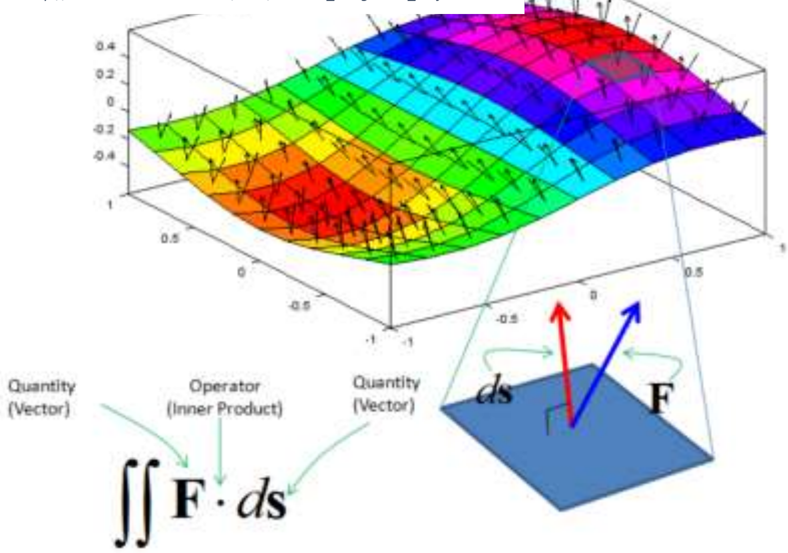
Area as a Vector:

The topic of vector areas traditionally causes considerable confusion for students. The starting point is the idea that we can use a vector to describe some of the properties of a surface. It's easiest to begin with a small (let's say rectangular) planar surface. The vector area $d\vec{S}$ we use to describe this surface is defined as being perpendicular to the surface and having a magnitude equal to the scalar area of the surface: $d\vec{S} = \hat{n}\, dS$. What is $\hat{n}$?

Let S be a two-sided surface of any shape, such as in Figure 18. Let one side of S be arbitrarily considered as the positive side (if S is a closed surface, such as a sphere, then the outer surface is the positive one). A unit normal $\hat{n}$ to any point of the positive side of S is called a *positive* or *outward drawn unit normal*.

Figure 37

http://www.sharetechnote.com/html/Calculus_Integration_Surface.html



In the figure, you see two vectors in each segments of the surface. One of the vectors is the said positive unit normal to each surface segment. The other one is in an arbitrary angle θ to the normal. Now I want to take the inner product of each red and blue vector and sum them all. This operation can be represented in the form of an integral. This, exactly, is a ***surface integral***:

$$\iint_S \vec{F}.d\vec{S} = \iint_S \vec{F}.\hat{n}\; dS = \iint_S F\cos\theta\; dS$$

Here $\iint_S$ denotes integration over the whole surface S. Do you remember the way we defined *Flux* in the ***Divergence*** section? That definition clearly tells you that the surface integral of a vector field over a surface signifies the total flux through that surface and *is a scalar quantity*.

Other types of surface integrals are $\iint_S \phi\; d\vec{S}$ and $\iint_S \vec{F} \times d\vec{S}$, which evidently are vector quantities.

e. Volume Integral

Say there is a closed surface enclosing a volume V and $\vec{A}$ is a single-valued and continuous vector function in that volume. Then the following denote the volume/space integral of that vector:

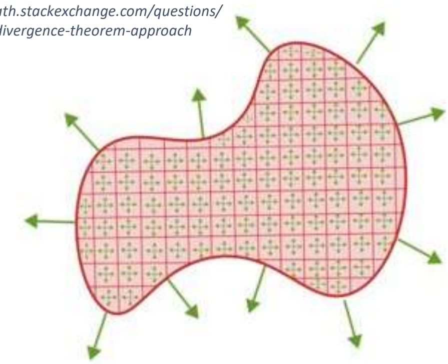
$$\iiint_V \vec{A}\; dV$$

f. Integral Theorems

Why were we studying these sorts derivative operators and integrals? The reason is that there are some very convenient and useful relations between them. We'll only study two of those here, but that's enough for now:

i. Gauss's Theorem / Divergence Theorem

Figure 39
https://math.stackexchange.com/questions/2277491/divergence-theorem-approach



The statement of the theorem is: *The surface integral of the normal component of a single-valued, continuous vector function over a closed surface is equal to the volume integral of the divergence of the vector over the volume enclosed by that same closed surface*. So, if S is a closed surface enclosing a volume element V, then

$$\iint_S \vec{A}.d\vec{S} = \iiint_V (\vec{\nabla}.\vec{A})\, dV$$

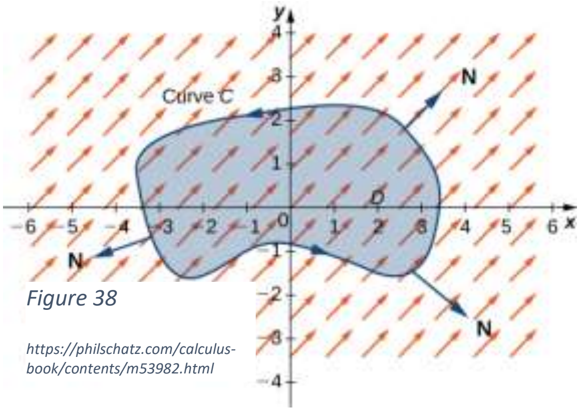
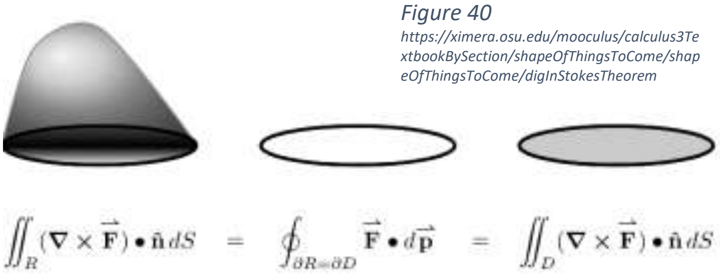


Figure 38
https://philschatz.com/calculus-book/contents/m53982.html

The significance? This theorem helps us convert a volume integral into a surface integral and vice-versa. Consider Figure 20, which shows the 2-D equivalent of the Gauss's theorem. See, using the theorem, we do not need to calculate the (cumbersome) volume integral if we can somehow express the integrand as the divergence of some vector. In terms of a vector field, as the surface integral depicts the total flux through the closed surface, and the total flux through the surface C in Figure 19 is zero (exactly same number of field lines come in and go out), the divergence of the vector field in the enclosed volume is also zero. This is exactly what we read for a solenoidal field in the ***Divergence*** section, right?

ii. Stokes' Theorem / Curl Theorem

Figure 40
https://ximera.osu.edu/mooculus/calculus3TextbookBySection/shapeOfThingsToCome/shapeOfThingsToCome/digInStokesTheorem



The statement of the theorem is: *The line integral of the tangential component of a well-defined vector field along a closed curve C is equal to the surface integral of the normal component of the curl of the vector over the surface enclosed by the same curve C.*

$$\oint_C \vec{A}.d\vec{l} = \iint_S (\vec{\nabla} \times \vec{A}).d\vec{S}$$

The beauty here is that the theorem does not say anything about the shape of the curve! It may be of any shape and size, but the integral only depends on the closed curve!

g. *Solved Questions*

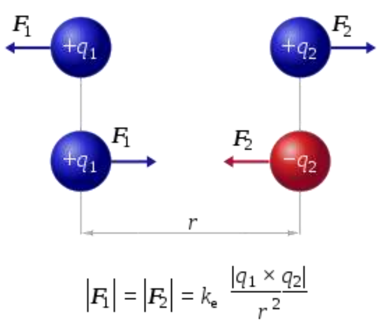
More problems to come…

Static Electricity (Electrostatics)-1

Contents

1.1 Coulomb's Law

***Point Charge**: A point charge is a hypothetical charge located at a single point in space (has no dimension).*



Coulomb's law, or *Coulomb's inverse-square law*, is an experimental law of physics that quantifies the amount of force between two stationary, electrically charged particles. The electric force between charged bodies at rest is conventionally called *electrostatic force* or *Coulomb force*. The quantity of electrostatic force between stationary charges is always described by Coulomb's law. The law was first published in 1785 by French physicist *Charles-Augustin de Coulomb*, and was essential to the development of the theory of electromagnetism, maybe even its starting point, because it made discussing quantity of electric charge possible in a meaningful way. (Source)

Statement: *The magnitude of the electric force between two stationary point charges is directly proportional to the magnitude of the charges and inversely proportional to the square of the distance between them.*

$$F_{vac} = \frac{1}{4\,\pi\epsilon_0}\frac{q_1 q_2}{r^2} = k_e\frac{q_1 q_2}{r^2},$$

where k_e is *Coulomb's constant* ($k_e \approx 8.9875517873681764 \times 10^9\ N\,m^2\,C^{-2}$), q_1 and q_2 are the signed magnitudes of the charges, and the scalar r is the distance between the charges. The force of the interaction between the charges is attractive if the charges have opposite signs (i.e., F_{vac} is negative) and repulsive if like-signed (i.e., F_{vac} is positive).

The physical constant ϵ_0 (pronounced as "epsilon nought" or "epsilon zero"), commonly called the *vacuum permittivity*, *permittivity of free space* or *dielectric constant* or the *distributed capacitance of the vacuum*, is the value of the absolute dielectric permittivity of classical vacuum. Its CODATA value is $\epsilon_0 = 8.8541878128(13) \times 10^{-12}\ F\,m^{-1}$ (*farads* per *metre*), with a relative uncertainty of 1.5×10^{-10}.

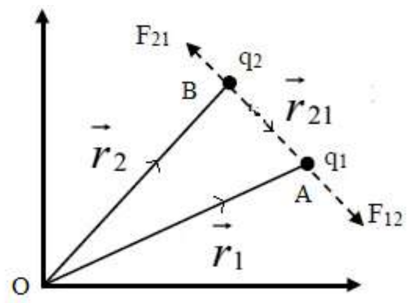
$$[\epsilon_0] = \frac{[Dim.of\ charge]^2}{[Dim.of\ Force][Dim.of\ Length]^2} = \frac{[I\ T]^2}{[M\ L\ T^{-2}][L]^2} = [M^{-1}L^{-3}T^4I^2]$$

This equation is true only for vacuum. For any other medium,

$$F_{med} = \frac{1}{4\,\pi\,\epsilon}\frac{q_1 q_2}{r^2}$$

where ϵ (*permittivity*) is a measure of the electric polarizability of a dielectric. A material with high *permittivity* polarizes more in response to an applied electric field than a material with low *permittivity*, thereby storing more energy in the electric field. In electrostatics, the permittivity plays an important role in determining the capacitance of a capacitor. The permittivity is often represented by the *relative permittivity* K, where $K = \frac{F_{vac}}{F_{med}} = \frac{\epsilon}{\epsilon_0}$. This is a dimensionless quantity.

Vector Form of Coulomb's Law: Let's say that there are two point charges $q_1$and q_2 placed at positions $\vec{r_1}$ and $\vec{r_2}$ respectively, w.r.t. the origin O of a Cartesian coordinate system. Position of q_2 w.r.t. q_1 is $\vec{r_{21}} = \vec{r_2} - \vec{r_1}$. Now, if the relative permittivity of the medium is ϵ_r, then the force on q_2 due to q_1 is



$$\vec{F_{21}} = \frac{1}{4\,\pi\,\epsilon_0\,\epsilon_r}\frac{q_1 q_2}{r_{12}^2}\,\hat{r}_{21},$$

where $\hat{r}_{21}$ is the unit vector from q_1 to q_2. This is the vector form of Coulomb's Law.

Now, the force exerted on q_1 due to q_2 is $\vec{F_{12}} = \frac{1}{4\,\pi\,\epsilon_0\,\epsilon_r}\frac{q_1 q_2}{r_{12}^2}\,\hat{r}_{12}.$

Here, $r_{12} = r_{21} = |\vec{r_{21}}| = |\vec{r_{12}}|$.

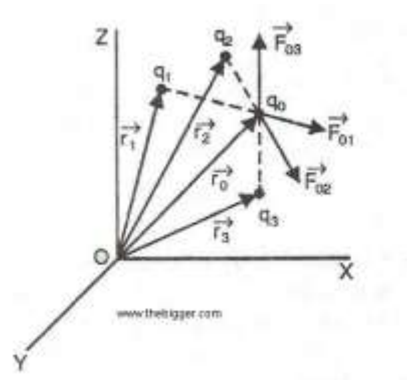
As $\hat{r}_{12} = -\hat{r}_{21}$, $\vec{F_{12}} = -\vec{F_{21}}$. From the adjacent figure, it is clear that $r_{12} = |\vec{r_{21}}| = |\vec{r_2} - \vec{r_1}|$. So

$$\vec{F_{12}} = \frac{1}{4\,\pi\,\epsilon_0\,\epsilon_r}\frac{q_1 q_2}{|\vec{r_2} - \vec{r_1}|^3}\,(\vec{r_2} - \vec{r_1}).$$

This clearly is a *Central Force* (acts along the line joining the centers of two charged bodies).

So $\vec{F} = \frac{q_1 q_2}{r^3}\,\vec{r} = f(r)\,\vec{r}$.

1.2 Law of Superposition

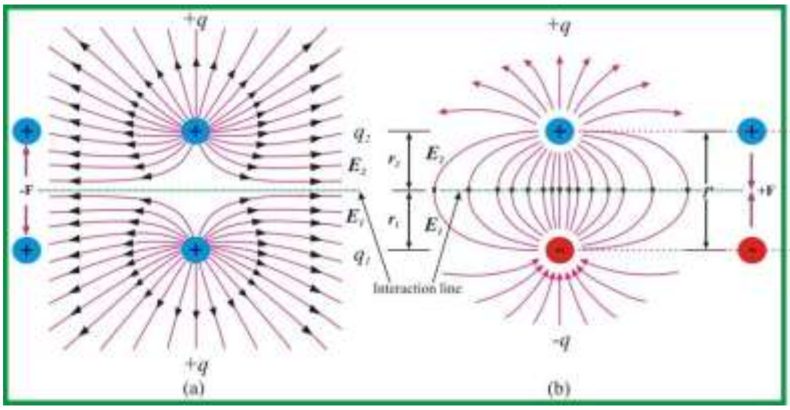


The *law of superposition* allows Coulomb's law to be extended to include any number of point charges. The force acting on a point charge due to a system of point charges is simply the vector addition of the individual forces acting alone on that point charge due to each one of the charges. The resulting force vector is parallel to the electric field vector at that point, with that point charge removed.

Now, the net electrostatic force $\vec{F}$ on a small test charge q_0 at position $\vec{r_0}$, due to a system of n discrete charges $(q_1, q_2, q_3, \ldots, q_n)$ at positions $\vec{r_1}, \vec{r_2}, \vec{r_3}, \ldots, \vec{r_n}$ in a medium of relative permittivity ϵ_r is

$$\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \cdots + \vec{F}_n = \sum_i \vec{F}_i = \frac{q_0}{4\,\pi\epsilon_0\epsilon_r}\sum_{i=1}^{n}\frac{q_i}{|\vec{r}-\vec{r_i}|^3}(\vec{r}-\vec{r_i})$$

1.3 Electrostatic Field



An electric field (sometimes abbreviated as "*E*" -field) surrounds an electric charge, and exerts force on other charges in the field, attracting or repelling them. These are created by electric charges, or by time-varying magnetic fields. When created by stationary charges, it is called *Electrostatic Field*. Electric fields and magnetic fields are both manifestations of the *electromagnetic force*, one of the 4 fundamental forces (or *interactions*) of nature. On an atomic scale, it is responsible for the attractive force between the atomic nucleus and electrons that holds atoms together, and the forces between atoms that cause chemical bonding.

Definition: It is defined mathematically as *a vector field that associates to each point in space the (electrostatic or Coulomb) force per unit of charge exerted on an infinitesimal positive test charge at rest at that point*.

The SI unit for electric field strength is *volt* per *meter* (V/m), exactly equivalent to *newton* per *coulomb* (N/C) in the SI system. $1\,NC^{-1} = \frac{1}{3\times10^4}\frac{dyne}{esu} = 1\frac{N\,m}{C\,m} = 1\frac{J}{C\,m} = 1\,V\,m^{-1}$.

We have seen from the last section that

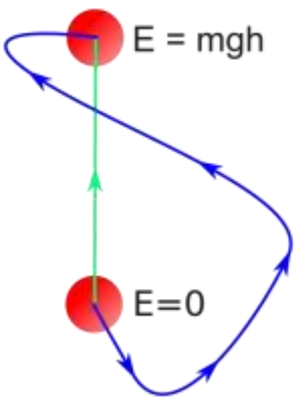
$$\vec{F} = \frac{q}{4\,\pi\epsilon_0\epsilon_r}\sum_{i=1}^{n}\frac{q_i}{|\vec{r}-\vec{r_i}|^3}(\vec{r}-\vec{r_i})$$

$$\therefore \vec{E} = \frac{\vec{F}}{q} = \frac{1}{4\,\pi\epsilon_0\epsilon_r}\sum_{i=1}^{n}\frac{q_i}{|\vec{r}-\vec{r_i}|^3}(\vec{r}-\vec{r_i})$$

$\therefore [E] = \frac{[F]}{[q]} = \frac{[M\,L\,T^{-2}]}{[I\,T]} = [M\,L\,T^{-3}\,I^{-1}]$.

1.4 Curl and Divergence of $\vec{E}$

1.4.1 Conservative Force



Definition: *A conservative force is a force with the property that the total work done in moving a particle between two points is independent of the taken path*. Gravitational force is an example of a conservative force, while frictional force is an example of a non-conservative force.

If a particle travels in a closed loop, the total work done (the sum of the force acting along the path multiplied by the displacement) by a conservative force is zero.

$$W = \oint \overrightarrow{F_{conv}}\,.d\vec{r} = 0$$

Now, $\oint \vec{F}\,.d\vec{r} = \iint (\vec{\nabla}\times\vec{F}).d\vec{s}$ (*Stokes' Theorem*).

$\therefore (\vec{\nabla}\times\vec{F}) = 0$ for conservative forces.

1.4.2 $\vec{\nabla}\times\vec{E}$ and $\vec{\nabla}.\vec{E}$

Let's check whether Electrostatic force is conservative or not:

$$\vec{F} = f(r)\,\vec{r} = f(r)\big(\hat{\imath}\,x + \hat{\jmath}\,y + \hat{k}\,z\big) = q_1q_2(\hat{\imath}\frac{x}{r^3} + \hat{\jmath}\frac{x}{r^3} + \hat{k}\frac{z}{r^3})$$

$$\therefore \vec{\nabla}\times\vec{F} = q_1q_2\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ \frac{x}{r^3} & \frac{y}{r^3} & \frac{z}{r^3}\end{vmatrix} = 0$$

Hence, electrostatic force is conservative as well as a central force (from earlier section). A *central force* is conservative if and only if it is spherically symmetric. This also means that

$$\vec{\nabla}\times\vec{E} = 0$$

A conservative force depends only on the position of the object. If a force is conservative, it is possible to assign a numerical value for the potential at any point and conversely, when an object moves from one location to another, the force changes the potential energy of the object by an amount that does not depend on the path taken, contributing to the mechanical energy and the overall conservation of energy. If the force is not conservative, then defining a scalar potential is not possible, because taking different paths would lead to conflicting potential differences between the start and end points.

$$\vec{F} = -\vec{\nabla}\phi$$

For a point charge,

$$\int_A^B \vec{F}.d\vec{r} = \frac{q}{4\pi\epsilon_0}\int_A^B \frac{1}{r^3}(x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}).(dx\,\hat{\imath} + dy\,\hat{\jmath} + dz\,\hat{k}) = \frac{q}{4\pi\epsilon_0}\int_A^B \frac{1}{r^3}(xdx + ydy + zdz)$$

$$= \frac{q}{4\pi\epsilon_0}\int_A^B \frac{rdr}{r^3} = \frac{q}{4\pi\epsilon_0}\int_A^B \frac{dr}{r^2} = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{r_A} - \frac{1}{r_B}\right]$$

Now,

$$\vec{\nabla}.\vec{E} = \frac{q}{4\pi\epsilon_0}\left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).\left[\hat{\imath}\frac{x}{r^3} + \hat{\jmath}\frac{y}{r^3} + \hat{k}\frac{z}{r^3}\right] = \frac{q}{4\pi\epsilon_0}\left[\frac{\partial}{\partial x}\left(\frac{x}{r^3}\right) + \frac{\partial}{\partial y}\left(\frac{y}{r^3}\right) + \frac{\partial}{\partial z}\left(\frac{z}{r^3}\right)\right]$$

$$= \frac{q}{4\pi\epsilon_0}\left[\frac{\partial}{\partial x}\left(\frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}}\right) + \frac{\partial}{\partial y}\left(\frac{y}{(x^2+y^2+z^2)^{\frac{3}{2}}}\right) + \frac{\partial}{\partial z}\left(\frac{z}{(x^2+y^2+z^2)^{\frac{3}{2}}}\right)\right]$$

$$= \frac{q}{4\pi\epsilon_0}\left[\frac{3}{r^3} - \frac{3}{r^3}\right] = 0 \qquad (x \neq 0, y \neq 0, z \neq 0)$$

This is true for all points in space other than the position of the charge ($r = 0$). There $\vec{E} \to \infty$, hence $\vec{\nabla}.\vec{E} \to 0$. Let's apply the *Divergence Theorem* to this:

$$\iiint (\vec{\nabla}.\vec{E})\, d^3r = \oint \vec{E}.d\vec{s} = \frac{1}{4\pi\epsilon_0}\int \left(\frac{q}{r^2}\hat{r}\right).(r^2 sin\theta\, d\theta\, d\phi\, \hat{r}) = \frac{q}{\epsilon_0} = \frac{1}{\epsilon_0}\iiint \rho\, d^3r\, \delta^3(\vec{r})$$

$$\Rightarrow \vec{\nabla}.\vec{E} = \frac{\rho}{\epsilon_0}\,\delta^3(\vec{r})$$

Here, ρ is charge density and $\int \delta^3(r)d^3r = 1$.

Remember, $\delta^3(r) \to \begin{cases} \infty, & r = 0 \\ 0, & r \neq 0 \end{cases}$.

1.5 Electrostatic Flux

1.5.1 Definition

In electromagnetism, electric flux is the measure of the electric field through a given surface, although an electric field in itself cannot flow. It is a way of describing the electric field strength at any distance from the charge causing the field. An electric "charge", such as a single electron in space, has an electric field surrounding it. In pictorial form, this electric field is shown as a dot, the charge, radiating "lines of flux". These are called *Gauss lines*. The density of these lines corresponds to the electric field strength, which could also be called the *electric flux density*: the number of "lines" per unit area. Electric flux is proportional to the total number of electric field lines going through a surface. For simplicity in calculations, it is often convenient to consider a surface perpendicular to the flux lines.

$$\Phi = \vec{E}.\vec{S} = \iint_S \vec{E}.d\vec{s}$$

If the electric field is uniform, the electric flux passing through a surface of vector area $\vec{S}$ is

$$\Phi = \vec{E}.\vec{S} = S\,E\,cos\theta$$

$E\,cos\theta$ is the perpendicular component of the electric field.

Dimension of flux: $[\Phi] = [M\,L^3T^{-3}\,I^{-1}]$

1.5.2 Numerical Examples

I. *Two point charges of* $+5C$ *and* $+15C$ *are at points* $(2,-4,3)m$ *and* $(-3,2,1)m$ *respectively. Find the force on the charge* $+15C$.

Answer:

$q_1 = +5C;$ $\qquad$ $q_2 = +15C$
$\vec{r_1} = 2\,\hat{\imath} - 4\,\hat{\jmath} + 3\,\hat{k};$ $\qquad$ $\vec{r_2} = -3\,\hat{\imath} + 2\,\hat{\jmath} + \hat{k}$
$\therefore \vec{r_2} - \vec{r_1} = -5\,\hat{\imath} + 6\,\hat{\jmath} - 2\,\hat{k}$
$\Rightarrow |\vec{r_2} - \vec{r_1}| = \sqrt{(-5)^2 + 6^2 + (-2)^2}\,m = \sqrt{65}\,m$
Hence, the force on the $+15C$ charge is

$$\overrightarrow{F_{21}} = \frac{1}{4\pi\varepsilon_0}\frac{5\times 15}{65^{3/2}}\left(-5\,\hat{\imath} + 6\,\hat{\jmath} - 2\,\hat{k}\right)N = \frac{9\times 10^9 \times 5 \times 15}{65^{\frac{3}{2}}}\left(-5\,\hat{\imath} + 6\,\hat{\jmath} - 2\,\hat{k}\right)N$$
$$= 1.288\times 10^9\left(-5\,\hat{\imath} + 6\,\hat{\jmath} - 2\,\hat{k}\right)N$$

II. *Show that the electric field* $\vec{E} = x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k}$ *is conservative.*

Answer:

$$\vec{\nabla}\times\vec{E} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = 0$$

Hence, the field $\vec{E}$ is conservative.

III. *If the field intensity at any point* (x,y,z) *is* $\vec{E} = 6xy\,\hat{\imath} + (3x^2 - 3y^2)\,\hat{\jmath} + 4z\,\hat{k}$, *what should be the amount of work done to take a positive charge from the origin to the point* (x_1, y_1, z_1)?

Answer:

Work done

$$= \int_{(0,0,0)}^{(x_1,y_1,z_1)} \vec{E}.d\vec{r} = \int_{(0,0,0)}^{(x_1,y_1,z_1)} 6xy\,dx + (3x^2 - 3y^2)\,dy + 4z\,dz$$
$$= [3x^2y + 3x^2y - y^3 + 2z^2]_{(0,0,0)}^{(x_1,y_1,z_1)} = 6x_1^2y_1 - y_1^3 + 2z_1^2$$

IV. *Find the electric flux through an area of 20 units in the YZ plane, for the uniform electric field* $\vec{E} = 6\,\hat{\imath} + 3\,\hat{\jmath} + 4\,\hat{k}$

Answer:

As the field is uniform, electric flux

$$\Phi_E = \iint \vec{E}.d\vec{s} = \vec{E}.\vec{s}$$

Now, $\vec{s} = 20\,\hat{\imath}\,sq\,units$
Hence, $\Phi_E = \left(6\,\hat{\imath} + 3\,\hat{\jmath} + 4\,\hat{k}\right).(20\,\hat{\imath}) = 120\,units$

V. *There is a charge of* $17.7\mu C$ *at the center of a spherical plane of radius* $5\ cm$. *What is the amount of the electric flux through the spherical plane?*

Answer:

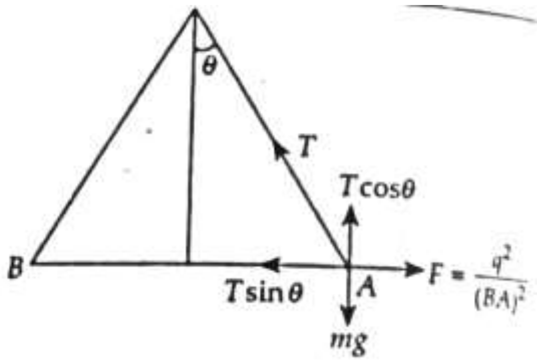
Area of a sphere of radius $r = 4\pi r^2$. Here $|\vec{E}| = \frac{q}{4\pi\varepsilon_0 r^2}$.

Hence, $\Phi_E = \vec{E}.d\vec{s} = \frac{q}{4\pi\varepsilon_0 r^2} \times 4\pi r^2 = \frac{q}{\varepsilon_0} = \frac{17.7\times10^{-6}C}{8.854\times10^{-12}C^2N^{-1}m^{-2}} = 2\times10^6 Nm^2C^{-1}$

We see that the flux is independent of the radius of the sphere. This is a consequence of the Coulomb's inverse-square law.

VI. *Two identical particles of mass m and charge q are dangling from the same point with two identical inextensible strings of length l. If each particle makes an angle θ with vertical, show that $4mgl^2 \sin^3\theta = q^2\cos\theta$*

Answer:



The adjacent figure depicts the stated problem.

According to the figure, $T\sin\theta = F = \frac{q^2}{(BA)^2}$;

Again, $BA = 2l\sin\theta$

$\Rightarrow T\sin^3\theta = \frac{q^2}{4l^2}$;

$T\cos\theta = mg$

We get, after removing T,

$4mgl^2\sin^3\theta = q^2\cos\theta$

VII. *Infinite number of point charges, each with charge q, are kept on the x-axis at points $x = 1, 2, 4, 8, \ldots$ etc. what would be the field intensity at $x = 0$ for these charges? If the charges are alternatively positive and negative, what would be the field then?*

Answer:

$$E = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{1^2}+\frac{q}{2^2}+\frac{q}{4^2}+\frac{q}{8^2}+\cdots\infty\right) = \frac{q}{4\pi\varepsilon_0}\sum_{n=0}^{\infty}(2^n)^{-2} = \frac{q}{4\pi\varepsilon_0}\sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^n$$

From the closed-form formula of infinite geometric series, we know,

$$a + ar + ar^2 + ar^3 + ar^4 + \cdots\infty = \sum_{n=0}^{\infty} a\,r^n = \frac{a}{1-r}, \qquad \text{for } |r| < 1$$

$$\Rightarrow 1 + r + r^2 + r^3 + r^4 + \cdots\infty = \frac{1}{1-r}$$

Hence

$E = \frac{q}{4\pi\varepsilon_0} \times \frac{1}{1-\frac{1}{4}} = \frac{q}{3\pi\varepsilon_0}$ (in the direction of negative x axis).

In the second case,

$$E = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{1^2}-\frac{q}{2^2}+\frac{q}{4^2}-\frac{q}{8^2}+\cdots\infty\right) = \frac{q}{4\pi\varepsilon_0}\left[\sum_{n=0}^{\infty}\left(\frac{1}{4^2}\right)^n - \sum_{n=0}^{\infty}\frac{1}{2^2}\left(\frac{1}{4^2}\right)^n\right]$$

$$= \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{1-\frac{1}{4^2}} - \frac{\frac{1}{2^2}}{1-\frac{1}{4^2}}\right] = \frac{q}{4\pi\varepsilon_0}\left[\frac{16}{15}-\frac{4}{15}\right] = \frac{q}{5\pi\varepsilon_0}.$$

If the closest charge is negative, then the direction of the field would be to the positive x axis.

VIII. *A square area parallel to the YZ plane is kept inside an electric field of intensity $(2\,\hat{\imath} + 3\,\hat{\jmath} + 5\,\hat{k})\,NC^{-1}$. The flux through the square is measured to be $8Nm^2C^{-1}$. What is the length of one side of the square?*

Answer:

$\vec{E} = (2\,\hat{\imath} + 3\,\hat{\jmath} + 5\,\hat{k})\,NC^{-1}$, $\vec{s} = a^2\,\hat{\imath}\,m^2$ (a = length of a side of the square)
Hence, $\vec{E}.\vec{s} = 2a^2Nm^2C^{-1} = 8\,Nm^2C^{-1}$
$\Rightarrow 2a^2 = 8 \Rightarrow a = 2$
i.e., length of a side of the square $= 2\,m$

IX. *Two point charges of magnitudes $+100\mu C^{-1}$ and $-100\mu C^{-1}$ are kept at two points A and B of an equilateral ABC with length of each side $= 10\ cm$. Find the Direction and magnitude of the electric field at point C due to these charges.*

Answer:

$F_1 = \frac{q}{4\pi\varepsilon_0 r^2} = \frac{100\times10^{-6}\times9\times10^{9}}{(10\times10^{-2})^2} = 9\times10^7 N$ (along $\overrightarrow{AC}$)
$F_2 = \frac{100\times10^{-6}\times9\times10^{9}}{(10\times10^{-2})^2} = 9\times10^7 N$ (along $\overrightarrow{CB}$)
As $F_1 = F_2$, their resultant bisects the angle $(180° - \angle BCA) = (180° - 60°) = 120°$.
Hence, the resultant $F = F_1 \cos\frac{120°}{2} + F_2 \cos\frac{120°}{2} = 9\times10^7 \times \left(\frac{1}{2}+\frac{1}{2}\right) = 9\times10^7\ NC^{-1}$
The resultant will be directed parallel to AB, in the direction from A to B.

1.6 Gauss's Theorem in Electrostatics:

1.6.1 Description

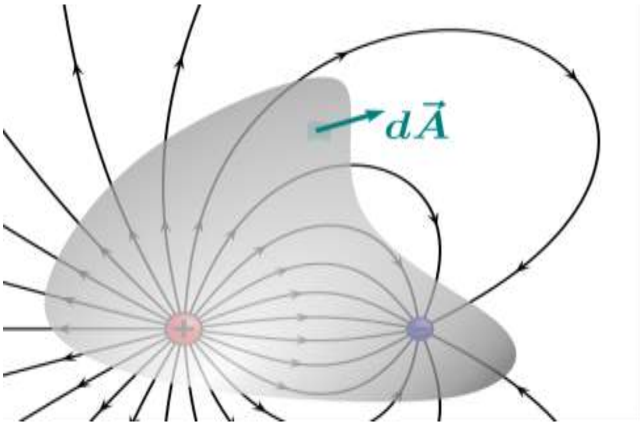


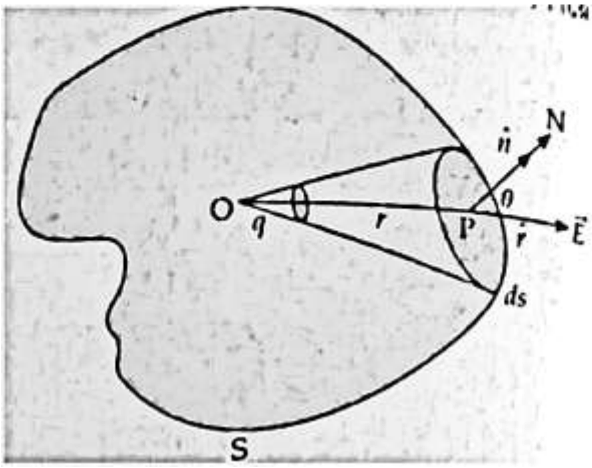
Figure 41 Electric flux through an arbitrary surface is proportional to the total charge enclosed by the surface.

Gauss's law/theorem, aka Gauss's flux theorem, relates a distribution of electric charge to the resulting electric field. First formulated by Joseph-Louis Lagrange (1773) and then by Carl Friedrich Gauss (1813), it is now one of Maxwell's four basic equations of classical electrodynamics.

Gauss's law can be used to derive Coulomb's law, and vice versa. There are two forms of the same law: the integral form and the differential form. These two forms are related through the *Divergence Theorem*. In words, the law states that:

*The **net electric flux** through any hypothetical **closed** surface is equal to $\frac{1}{\varepsilon_0}$ times the **net electric charge** within that closed surface, where ε_0 is the absolute dielectric permittivity of the classical vacuum.*

A closed surface is a surface that is compact and without boundary. An easy way to remember a closed surface is that if one wants to go from one side of the surface to the other side, one *has to* pass through the surface (there's no way around it).
The flux here is proportional to the enclosed electric charge, irrespective of how that charge is distributed. Even though the law alone is insufficient to determine the electric field across a surface enclosing any charge distribution, this may be possible in cases of uniform fields – the uniformity demanded by some symmetry. Where no such symmetry exists, Gauss's law can be used in its differential form.



Proof: Let us take a closed surface S of any shape. There is a charge q at the point O inside this surface. Let us now consider an infinitesimal part of the surface dS around the point P, where PN is perpendicular to dS. The electric field at P due to q at O is $\vec{E} = \frac{1}{4\pi\varepsilon_0}.\frac{q}{r^2}\hat{r}$ and it acts along $\overrightarrow{OP}$. The solid angle subtended at point O by the area dS is $d\Omega$. θ is the angle between $\overrightarrow{ON}$ and $\overrightarrow{OP}$, i.e. the direction of the electric field at P. Hence, the electric flux passing through the area dS is

$$d\phi = \vec{E}.d\vec{S} = \left(\frac{1}{4\pi\varepsilon_0}.\frac{q}{r^2}\hat{r}\right).(\hat{n}dS)$$
$$= \frac{1}{4\pi\varepsilon_0}.\frac{q}{r^2}\,(\hat{r}.\hat{n})$$
$$= \frac{1}{4\pi\varepsilon_0}.\frac{q}{r^2}\,dS\cos\theta$$

$[\because \hat{r}.\hat{n} = \cos\theta]$
Hence, total flux through the closed surface S,

$$\phi = \oiint_S \vec{E}.\vec{dS} = \oiint_S \frac{1}{4\pi\varepsilon_0}.\frac{q}{r^2}\,dS\cos\theta = \frac{q}{4\pi\varepsilon_0}\oiint_S \frac{dS\cos\theta}{r^2} = \frac{q}{4\pi\varepsilon_0}\oiint_S d\Omega = \frac{q}{4\pi\varepsilon_0}\times 4\pi$$

$$= \frac{q}{\varepsilon_0} \qquad \left[\because \oiint_S d\Omega = 4\pi\right]$$

Hence, the total flux passing through a closed surface enclosing a charge q is $\frac{1}{\varepsilon_0}$ times the amount of charge, i.e. q. If there are multiple point charges, the total flux would be proportional to the algebraic sum of those charges:

$$\Phi_E = \oiint \vec{E}.\vec{dS} = \frac{1}{\varepsilon_0}\sum_i q_i$$

Some notes about Gauss's theorem:

1. For an electric dipole, $\sum_i q_i = 0$, and hence, the flux due to a electric dipole $= 0$.
2. This theorem tells us that if there is no charge within the closed surface then flux through that surface is zero. Zero flux does not always indicate zero electric field. For example, though flux due to a dipole is zero, electric field due to that is non-zero.
3. We will see later that we can calculate the electric field at any point using Gauss's theorem, if only the charge distribution has some kind symmetry to it. This is not possible from Coulomb's law.

1.6.2 Differential Form

If ρ is the volume charge density at a region of space, then the flux around a closed surface S enclosing the charged volume V is, from Gauss's theorem,

$$\Phi = \oiint_S \vec{E}.\overrightarrow{dS} = \frac{q}{\varepsilon_0} = \frac{1}{\varepsilon_0}\iiint_V \rho\, d^3r\,.$$

Now, from divergence theorem, $\oiint_S \vec{E}.\overrightarrow{dS} = \iiint_V (\vec{\nabla}.\vec{E})\, d^3r$.

Hence, $\iiint_V \left(\vec{\nabla}.\vec{E} - \frac{\rho}{\varepsilon_0}\right) d^3r = 0$. This is true for any arbitrary V and S. So, we can write,

$$\vec{\nabla}.\vec{E} = \frac{\rho}{\varepsilon_0}$$

This is the differential form of Gauss's theorem. In words, the divergence of an electric field at any point is equal to the ratio of the volume charge density at that point and the permittivity of vacuum. This is also known as the first of the *Maxwell's Equations of Electrodynamics*.

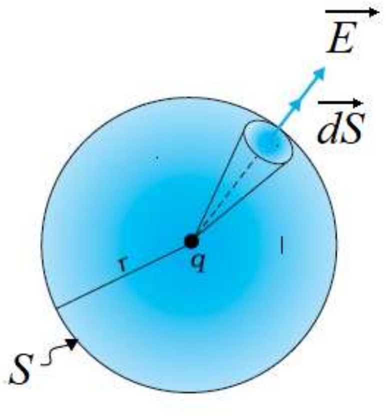
If we define $\vec{D} = \varepsilon_0\vec{E}$, the law becomes $\vec{\nabla}.\vec{D} = \rho$.

As the electric field $\vec{E}$ is conservative, $\vec{\nabla}\times\vec{E} = 0$. We can thus write $\vec{E}$ as the gradient of a scalar potential ϕ: $\vec{E} = -\vec{\nabla}\phi$.
Hence, $\vec{\nabla}.\vec{E} = \frac{\rho}{\varepsilon_0} \Rightarrow \vec{\nabla}.\vec{\nabla}\phi = -\frac{\rho}{\varepsilon_0}$. Or,

$$\nabla^2\phi = -\frac{\rho}{\varepsilon_0}$$

This equation is known as the Poisson's equation. For a region which is electric charge free, this becomes $\nabla^2\phi = 0$: this is known as Laplace's equation.

1.6.3 Coulomb's Law from Gauss's Theorem



Coulomb's law cannot be derived using only Gauss's law, since Gauss's law does not give any information regarding $\vec{\nabla}\times\vec{E}$. However, Coulomb's law is easy to prove from Gauss's law in presence of additional assumption that the electric field from a point charge is spherically symmetric (this assumption, like Coulomb's law itself, is exactly true if the charge is stationary, and approximately true if the charge is in motion).
Taking S in the integral form of Gauss's law to be a spherical surface of radius r, centered at the point charge q, we have,

$$\oiint_S \vec{E}.\overrightarrow{dS} = \frac{q}{\varepsilon_0} \Rightarrow E\oiint_S \overrightarrow{dS} = \frac{q}{\varepsilon_0}$$

Here $\vec{E}.\overrightarrow{dS} = E\,dS$, as the magnitude of $\vec{E}$ is constant on every point of the spherical surface, due to symmetry and both $\vec{E}$ and $\overrightarrow{dS}$ are directed radially outward, i.e., have the same direction.

As $\oiint_S \overrightarrow{dS} = 4\pi r^2$,

$$E\times 4\pi r^2 = \frac{q}{\varepsilon_0} \Rightarrow E = \frac{q}{4\pi\varepsilon_0 r^2}$$

Now, if we put another point charge q_0 at the same distance r from q, the force acting on q_0 due to q is,

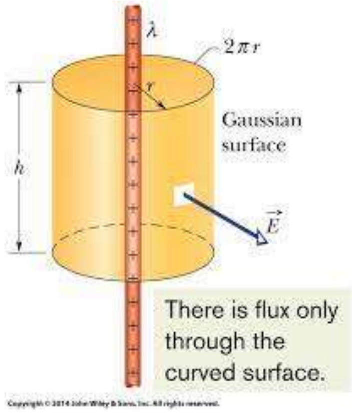
$$F = E\,q_0 = \frac{q\,q_0}{4\pi\varepsilon_0 r^2}$$

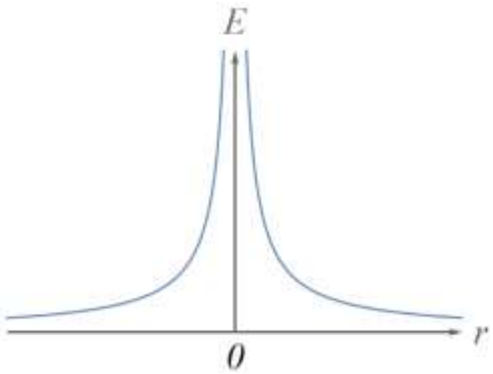
1.6.4 Applications of Gauss's Theorem

I. *Electric field due to a* ***point charge***: The description is already there in the last subsection.

$$\vec{E} = E\hat{r} = \frac{q}{4\pi\varepsilon_0 r^2}\hat{r} = \frac{q\,\vec{r}}{4\pi\varepsilon_0 r^3}$$

II. *Electric Field due to a uniformly charged* ***long straight wire***: As the wire is uniformly charged, let's say that the charge at each unit length of the wire, i.e., the linear charge density is λ. The job is to calculate the electric field due to this wire at a distance r from the wire. If we imagine a cylindrical Gaussian surface of length h around the straight wire, we find that due to cylindrical symmetry of the wire, electric flux lines will only pass through the curved Gaussian surface. Hence, the total electric flux through the surface:





$$\oint\!\!\oint_S \vec{E}.\overrightarrow{dS} = E\,(2\pi r h)$$

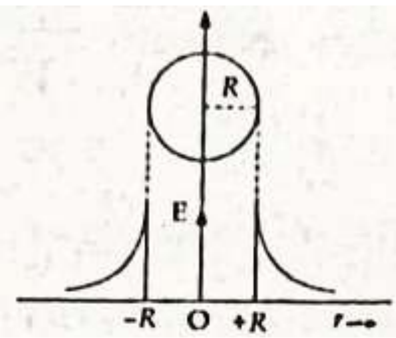
Now, the total charge enclosed by the Gaussian surface is λh. From Gauss's theorem,

$$E\,(2\pi r h) = \frac{\lambda h}{\varepsilon_0} \Rightarrow E = \frac{1}{4\pi\varepsilon_0}\frac{2\lambda}{r}$$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{2\lambda}{r}\hat{r}$$

The direction of $\vec{E}$ is radially outward.

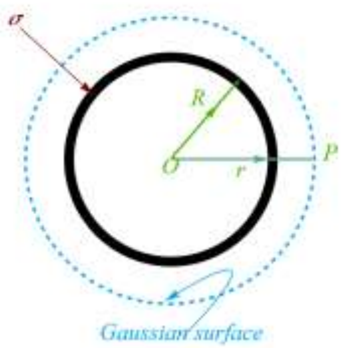
III. *Electric Field due to a uniformly charged spherical shell*: Let's say that we have a (very) thin spherical shell of radius R and center O. The surface of the spherical shell is uniformly charged with surface-charge-density σ. So, the total charge-content of the shell is $q = 4\pi R^2\sigma$. The job is to find electric field due to q at the point P at a distance r from the center of the shell O. There are three cases here, depending on the relative sizes of R and r. The figure on the left shows the change in the electric field with change in r. The reasons for this particular nature of E vs. r is discussed below.

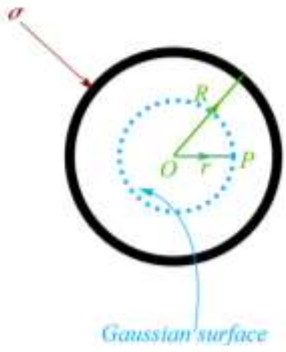


a. At a point outside the shell ($r > R$): If we imagine a spherical Gaussian surface of radius r with center at O, then the electric field at any point on this surface is same and the field lines cut the surface at right angle. Hence, the total electric flus through this surface, $\oint\!\!\oint_S \vec{E}.\overrightarrow{dS} = E(4\pi r^2)$. From Gauss's theorem,

$$E(4\pi r^2) = \frac{q}{\varepsilon_0} = \frac{4\pi R^2\sigma}{\varepsilon_0} \Rightarrow E = \frac{R^2\sigma}{r^2\varepsilon_0}$$

$$\Rightarrow \vec{E} = \frac{R^2\sigma}{\varepsilon_0 r^2}\hat{r}$$





b. At a point on the surface of the shell ($r = R$): Extending the relations from the above case,

$$\vec{E} = \frac{R^2\sigma}{\varepsilon_0 r^2}\hat{r} = \frac{\sigma}{\varepsilon_0}\hat{r}$$

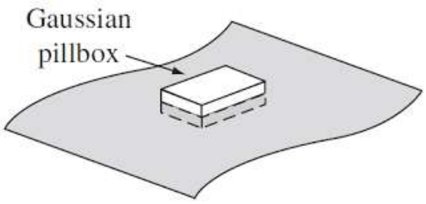
This is consistent with the observation that the electric field is spherically symmetric.

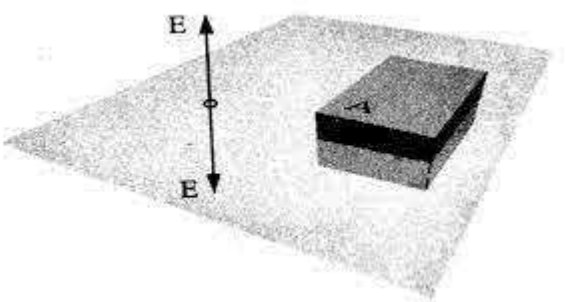
c. At a point inside the shell ($r < R$): In this case, the Gaussian surface lies completely within the shell. The whole amount of charge now is completely outside the Gaussian surface (as it resides on the surface). So, the charge enclosed by the Gaussian surface, $q = 0$. So, from Gauss's theorem,

$$\oiint_S \vec{E}.\overrightarrow{dS} = \frac{q}{\varepsilon_0} = 0$$

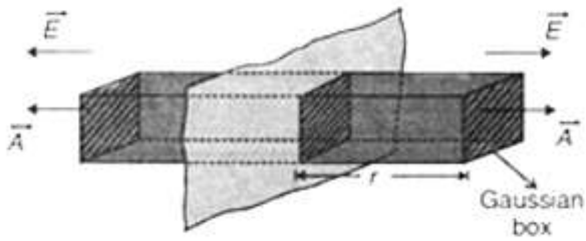
IV. *Electric field due to a uniformly charged infinite plane surface*:

Before we get into this, let us clarify something. It is a common practice in physics community to call what we are going to encounter here as a *'Gaussian Pill-box'*. A traditional pill-box may look like the one on the right side, or it may be circular, hexagonal or any other shape. That is not important for us. The only important general characteristic of a pill-box for us is that its height is much smaller than the area of its lid. That is it. Sometimes we will consider a closed Gaussian surface shaped like a pill-box, half-embedded in an actual surface, containing electric charge, like in the figure on the left.





In our present problem, we imagine an infinite charged flat plane with surface-charge-density σ. We draw an (imaginary) Gaussian pill-box at any place. The surface area of the pill-box is A, and the depth is $2r$. For ease of understanding, consider the surface A to be a square. Remember, following the definition of a pill-box, $r \ll \sqrt{A}$. As we see in the figure on the left, as the infinite plane is uniformly charged, all the field lines ($\vec{E}$) extend perpendicularly on both sides of the plane. In the figure on the right, we now zoom in to the pill-box and deliberately blow-up/stretch the pill-box for our understanding. It should be clear that the total charge enclosed by the whole pill-box is due to the cross-section of the plane by the pill-box, i.e., $q = \sigma A$. Using Gauss's theorem, we can write,



$$\oiint_S \vec{E}.\overrightarrow{dS} = \frac{q}{\varepsilon_0} = \frac{\sigma A}{\varepsilon_0}$$

For all surfaces of the pill-box that are perpendicular to the infinite plane, $\vec{E}.\overrightarrow{\Delta S} = 0$, where $\overrightarrow{\Delta S}$ is the area of any of those faces (area-vector of an area is normal to the area itself; hence $\vec{E} \perp \overrightarrow{\Delta S}$). If we call the parallel areas, respectively on the right and on the left as S_1 and S_2, and the right side as positive direction, then $S_1 = -S_2 = \vec{A}$. Thus, Gauss's theorem becomes,

$$\oiint_S \vec{E}.\overrightarrow{dS} = \iint_{S_1} \vec{E}.\overrightarrow{dS} + \iint_{S_2} (-\vec{E}).\overrightarrow{dS} + 4\iint_{\Delta S} \vec{E}.\overrightarrow{dS} = \frac{\sigma A}{\varepsilon_0}$$

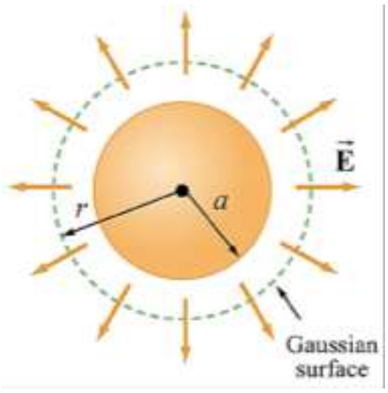
$$\Rightarrow \vec{E}.\vec{A} + (-\vec{E}).(-\vec{A}) + 4\times 0 = \frac{\sigma A}{\varepsilon_0} \Rightarrow 2|\vec{E}|A = \frac{\sigma A}{\varepsilon_0}$$

$$\Rightarrow |\vec{E}| = \frac{\sigma}{2\varepsilon_0} \Rightarrow \vec{E} = \frac{\sigma}{2\varepsilon_0}\hat{n}$$

where $\hat{n}$ is the unit-normal vector of the infinite plane.

V. *Electric field due to a uniformly charged solid sphere*:

Let us consider that the solid sphere in consideration is uniformly charged, with volume charge-density ρ. Just like the case of the spherical shell, if the point P at which we measure the electric field is at a distance r from the center of the solid sphere and the radius of the sphere is a, then we have three case, just as earlier,



a. $\boldsymbol{r > a}$: Here P is outside the sphere, at a distance r. The electric field due to the sphere points radially outward (i.e. perpendicular to the Gaussian surface at every point), and from spherical symmetry, is of constant magnitude on every point of the surface at r. So, the total amount of flux passing through the Gaussian surface,

$$\Phi_E = \oiint_S \vec{E}.\overrightarrow{dS} = |\vec{E}| \oiint_S \overrightarrow{dS} = 4\pi r^2 |\vec{E}|$$

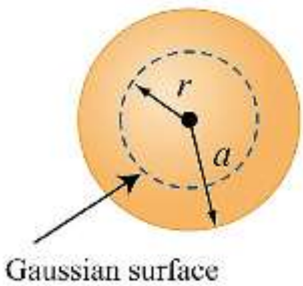
$$\Rightarrow 4\pi r^2 |\vec{E}| = \frac{q}{\varepsilon_0} = \frac{1}{\varepsilon_0} \times \frac{4}{3}\pi a^3 \rho$$

$$\Rightarrow |\vec{E}| = \frac{\rho}{3\varepsilon_0}\frac{a^3}{r^2}, \quad \text{(radially outward)}$$

Or, in other words, outside the sphere, $E \propto 1/r^2$.

b. $\boldsymbol{r = a}$: With a simple extension of the case above, it is trivial to show that when $r = a$, $|\vec{E}| = \frac{\rho}{3\varepsilon_0}a$.

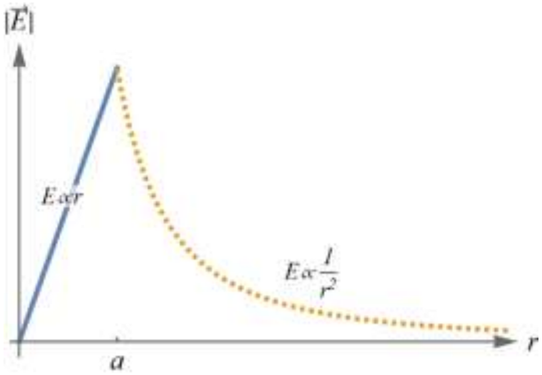
c. $\boldsymbol{r < a}$: Just like in the case of the spherical shell, we draw the Gaussian surface inside the sphere. Now, the charge enclosed by the Gaussian surface is



$$q' = \frac{4}{3}\pi r^3 \rho.$$

Using $q = \frac{4}{3}\pi a^3 \rho$, we see that $q' = \frac{qr^3}{a^3}$.

Now, from Gauss's theorem,



$$\oiint_S \vec{E}.\overrightarrow{dS} = \frac{q'}{\varepsilon_0} = \frac{qr^3}{\varepsilon_0 a^3} \Rightarrow 4\pi r^2 |\vec{E}|$$

$$= \frac{1}{\varepsilon_0}\frac{4}{3}\pi r^3 \rho$$

$$\Rightarrow |\vec{E}| = \frac{\rho r}{3\varepsilon_0}$$

Or, in other words, inside the sphere, $E \propto r$.

1.7 Electric Potential:

1.7.1 Potential as line integral of Field:

If the electric field at a point is $\vec{E}$, then the force on a test charge q_0 due to that electric field is $\vec{F} = q_0\vec{E}$. If the charge moves by an infinitesimal amount of $\overrightarrow{dl}$ because of the action of that force, then the work done by the electric field on that charge, $dW = \vec{F}.\overrightarrow{dl} = q_0\vec{E}.\overrightarrow{dl}$. Work done by the field to displace the charge from point A to point B is $W = \int_A^B dW = q_0 \int_A^B \vec{E}.\overrightarrow{dl}$. Hence the work done per unit charge by the electric field is $\frac{W}{q_0} = \int_A^B \vec{E}.\overrightarrow{dl} =$ the line integral of the electric field from point A to point B.

∴The work done by an electric field form one point to another is equal to the line integral of the electric field between those two points.

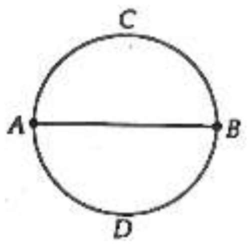
We know that a static electric field is conservative, i.e., $\vec{\nabla} \times \vec{E} = 0$. As $\vec{\nabla} \times \vec{\nabla}\phi = 0$ is an identity for any scalar function ϕ, we can write $\vec{E}$ as a gradient of a scalar function, i.e., $\vec{E} = -\vec{\nabla}\phi$. The negative sign is there so that we can define the potential with a correct sign. We will soon understand the meaning of the negative sign.

Now, if $\overrightarrow{dr} = (dx\,\hat{\imath} + dy\,\hat{\jmath} + dz\,\hat{k})$ and $\vec{\nabla}\phi = \left(\frac{\partial\phi}{\partial x}\hat{\imath} + \frac{\partial\phi}{\partial y}\hat{\jmath} + \frac{\partial\phi}{\partial z}\hat{k}\right)$,

$$\therefore \int_A^B \vec{E}.\overrightarrow{dr} = -\int_A^B \vec{\nabla}\phi.\overrightarrow{dr} = -\int_A^B \left(\frac{\partial\phi}{\partial x}\hat{\imath} + \frac{\partial\phi}{\partial y}\hat{\jmath} + \frac{\partial\phi}{\partial z}\hat{k}\right).(dx\,\hat{\imath} + dy\,\hat{\jmath} + dz\,\hat{k})$$

$$= -\int_A^B \left(\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz\right) = -\int_A^B d\phi$$

$$\Rightarrow \int_A^B \vec{E}.\overrightarrow{dr} = \phi_A - \phi_B$$



Where $\phi_P =$ electric potential of the electric field at a point P. Thus, the line integral of a static electric field depends only on the end points of the path of integration and is independent of the path itself.
For a closed path, as shown in the figure (left), we can see that,

$$\oint \vec{E}.\overrightarrow{dr} = \underset{(\text{along } ACB)}{\int_A^B} \vec{E}.\overrightarrow{dr} + \underset{(\text{along } BDA)}{\int_B^A} \vec{E}.\overrightarrow{dr} = \phi_A - \phi_B + \phi_B - \phi_A = 0$$

The next obvious question would be, what exactly *is* this potential? Think about gravity. Unlike the force between a positive charge and another positive test charge, gravity is always attractive. If we release a ball from point A (a higher point from ground), gravity acts on it and brings it down to (a lower point) B. The work done by gravitational field is then the different between the gravitational potentials between those points $(\phi_A^g - \phi_B^g)$. To raise a mass against gravity (from B to A), we need to work against gravitational field. We can, in principle, define a zero potential plane (most cases, the ground) relative to which the potential for every higher point is defined.

For a repulsive force (such as between two positive charges), assume that the source charge/charge distribution of the electric field is already there. Now to bring another positive test charge closer to it (from B to A), we have to work against the electric field. But didn't we have to work already against the field to bring that test charge to the point B? Sure! That is why point B also has a potential. It seems common sense that there will no effect of the field at infinity (∞), and we can use infinity as our zero potential point. We can then define the potential of any electric field of any point as *the work done to bring a unit charge from infinity to that point*. From earlier, work done by the field,

$$\int_{r}^{\infty} \vec{E}.\overrightarrow{dr} = \phi_r - 0 = \phi_r$$

$$\Rightarrow \phi_r = -\int_{\infty}^{r} \vec{E}.\overrightarrow{dr}$$

The negative sign denotes that here, work is done *against* the electric field to bring the charge *from* infinity to r.

Similarly, the potential difference between two points, $\phi_B - \phi_A = -\int_A^B \vec{E}.\overrightarrow{dr}$

Dimension of electric potential = $\frac{\text{dim.of work}}{\text{dim.of charge}} = \frac{[ML^2T^{-2}]}{[IT]} = [ML^2T^{-3}I^{-1}]$

More Material:

1) Fact Factor Site
2) Dimensions of S.I units and quantities.
3) Bozeman Science (Video)
4) Lecture by Walter Lewin. (Video)

Magnetism

Contents

2 Magnetic Fields and Lorentz Force

We have seen in the previous chapter that a static charge generates static electric field, but not an electric current. This is why there is no force acting on a static charge near a current-carrying conductor (wire). On the other hand, a moving charge in the same place experiences a force (keeping the charge static but the conductor moving produces a similar effect). This lets us hypothesize that there is a type of 'field' being generated by the current-carrying conductor – which only affects a charge in relative motion with respect to the conductor. Precise measurements prove this hypothesis and we call this field a *Magnetic field*. Just as the electric fields are conventionally denoted as $\vec{E}$, magnetic fields are denoted as $\vec{B}$. The net force acting on the charge in this case by both the electric and magnetic fields is called a *Lorentz force*. The law was implicit in a paper by J. C. Maxwell (1865), though later Oliver Heaviside correctly identified the contribution of the magnetic force and H. Lorentz finished a complete derivation with the electric force a few years later (1895).

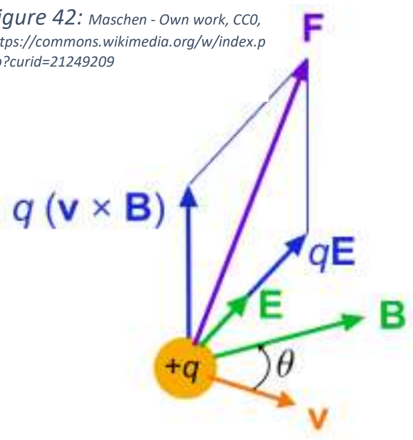


Figure 42: Maschen - Own work, CC0, https://commons.wikimedia.org/w/index.php?curid=21249209

The electromagnetic force $\vec{F}$ on a test charge q with a velocity $\vec{v}$ at a given point and time can be parameterized by exactly two vectors $\vec{E}$ and $\vec{B}$, in the functional form:

$$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

Clearly, in absence of the electric field, the force is perpendicular to both $\vec{v}$ and $\vec{B}$ and its direction is determined by the right-hand cork-screw rule of cross-product. As

$$F = \left|\vec{F}\right| = qBv\sin\theta \Rightarrow B = \frac{F}{qv\sin\theta}$$

$$\Rightarrow [B] = \frac{[MLT^{-2}]}{[IT][LT^{-1}]} = [MT^{-2}I^{-1}]$$

The units of a magnetic field are respectively *Tesla* (magnetic field which, when applied to an $1C$ charge, moving at $1m/s$ perpendicular to the field, exerts a force of $1N$ on the charge) and *Gauss* (magnetic field which, when applied to an $1emu$ charge, moving at $1cm/s$ perpendicular to the field, exerts a force of $1dyne$ on the charge) in SI and CGS units.

$$1T = \frac{1N}{1C \times 1ms^{-1}} = \frac{10^5 dyne}{0.1emu \times 100cms^{-1}} = 10^4 \frac{dyne}{emu \times cms^{-1}} = 10^4 G$$

3 Biot-Savart-Law

3.1 Definition (pronounced: '**by**-oh-*suh*-**vahr**')

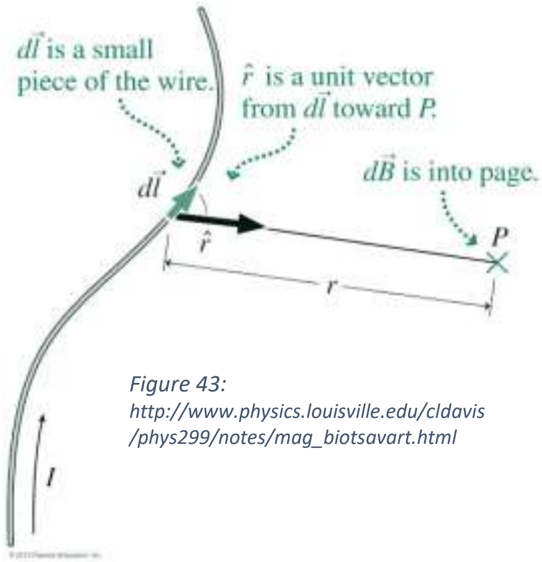
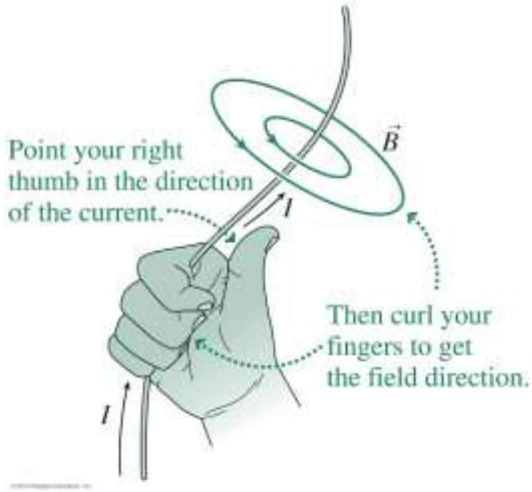


Figure 43: http://www.physics.louisville.edu/cldavis/phys299/notes/mag_biotsavart.html

We came to know in the last section that an electric conductor creates a magnetic field. The next job would be to know the *how* and *how much* of it. Named after *Jean-Baptiste Biot* and *Félix Savart* (1820), the Biot-Savart Law describes the magnetic field generated by a ***steady*** electric current in terms of the magnitude, direction, length, and proximity of the electric current.
A *steady* (or *stationary*) current is a continual flow of charges which does not change with time and the charge neither accumulates nor depletes at any point. Unlike electric charges, *there is nothing like a 'point current'*. Thus the law has to be written in a differential form, in terms of an infinitesimal *current element*. As shown in the adjacent figure, for a steady current I passing through a conductor, the current element corresponding the length element $d\vec{l}$ of the conductor is $Id\vec{l}$. If the infinitesimal magnetic field is $d\vec{B}$ due to the infinitesimal current element $d\vec{l}$ at a point P, $\vec{r}$ distance from the current element (where $\vec{r}$ makes the angle θ with $d\vec{l}$), then,

$$\left.\begin{array}{l} |d\vec{B}| \propto I|d\vec{l}| \\ |d\vec{B}| \propto \sin\theta \\ |d\vec{B}| \propto \frac{1}{|\vec{r}|^2} \end{array}\right\} |d\vec{B}| \propto \frac{I|d\vec{l}|\sin\theta}{|\vec{r}|^2}$$

$$\Rightarrow |d\vec{B}| = k\frac{I|d\vec{l}|\sin\theta}{|\vec{r}|^2}$$



k is a proportionality constant, determined by the unit of current. In SI system, $k = \frac{\mu_0}{4\pi}$, where $\mu_0 = (4\pi \times 10^{-7})Hm^{-1} = (4\pi \times 10^{-7})NA^{-2}$ is the magnetic permeability of the classical vacuum. The direction of this infinitesimal magnetic field is guided by the right-hand cork-screw rule. This means that at the point P (check Figure 2), the direction of the field is perpendicular to this page and into the page. This is generally denoted by $\otimes$ in literature (check the figure). Similarly, in figures where the field is perpendicularly coming out of the page, the usual symbol is $\odot$.

To remember the direction of the magnetic field, use the mnemonic in the figure above. With all these info at our disposal, we can now write the vectorial form of the equation:

$$d\vec{B} = \frac{\mu_0}{4\pi}\left(\frac{Id\vec{l} \times \hat{r}}{|\vec{r}|^2}\right) = \frac{\mu_0}{4\pi}\left(\frac{Id\vec{l} \times \vec{r}}{|\vec{r}|^3}\right)$$

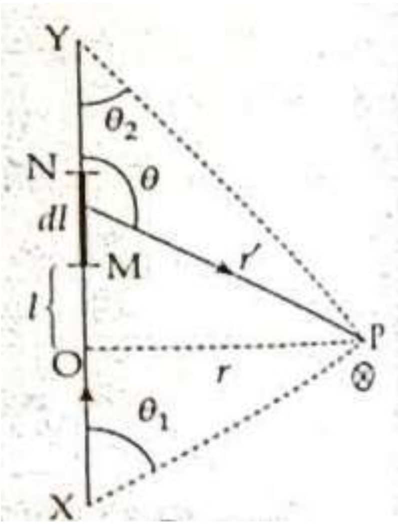
$$\Rightarrow \vec{B} = \frac{\mu_0}{4\pi}\int_C \left(\frac{Id\vec{l} \times \vec{r}}{|\vec{r}|^3}\right)$$

→ This is the total field generated due to the whole conductor. Some important points:

- When $\theta = 0°$, $d\vec{B} = 0$, i.e., the intensity of the magnetic field at all points on the axis of the current element is zero.
- When $\theta = 90°$, $d\vec{B}$ is maximum, i.e., the intensity of the magnetic field at all points lying perpendicular to the axis of the current element is maximum.
- The intensity mentioned above is called the magnetic induction vector or the magnetic flux density.

3.2 Applications of Biot-Savart Law:

a. Magnetic field intensity at a point due to a straight conductor:



Following the adjacent figure, XY is a straight conductor, through which electric current I is passing from X to Y. The job is to find out the magnetic field due to this wire at a point P. Just like in the previous section, direction of magnetic field at P would be into the page, i.e., ⊗. Let us say that the point at which the normal from P XY cuts the conductor is named O. $OP = r$. Imagine an infinitesimal part dl (MN) of the conductor at a distance l from O. Magnetic field at the point P due to the current element Idl is,

$$dB = \frac{\mu_0}{4\pi}\frac{Idl\sin\theta}{r'^2}$$

Now,

$$r = r'\sin(\angle P(dl)O) = r'\sin(\pi - \theta) = r'\sin\theta$$
$$\Rightarrow r' = r\csc\theta$$

Also,

$$r = l\tan(\angle P(dl)O) = l\tan(\pi - \theta) = -l\tan\theta$$
$$\Rightarrow l = -r\cot\theta \Rightarrow dl = r\csc^2\theta \ d\theta$$
$$\therefore dB = \frac{\mu_0}{4\pi}\frac{Idl\sin\theta}{r'^2} = \frac{\mu_0}{4\pi}\frac{I(r\csc^2\theta\, d\theta)\sin\theta}{r^2\csc^2\theta} = \frac{\mu_0}{4\pi}\frac{I}{r}\sin\theta\, d\theta$$

Hence, for the whole conductor XY, the field at P is,

$$B = \int dB = \frac{\mu_0}{4\pi}\frac{I}{r}\int_{\theta_1}^{\pi-\theta_2}\sin\theta\, d\theta = \frac{\mu_0}{4\pi}\frac{I}{r}[-\cos\theta]_{\theta_1}^{\pi-\theta_2} = \frac{\mu_0}{4\pi}\frac{I}{r}(\cos\theta_1 + \cos\theta_2)$$

Special Cases:

A. When the conductor is infinitely long, it spreads both directions in such a way that $\theta_1 = \theta_2 = 0°$. In that case, intensity of the magnetic field at P,

$$B = \frac{\mu_0}{4\pi}\frac{I}{r}(\cos 0° + \cos 0°) = \frac{\mu_0}{2\pi}\frac{I}{r}$$

B. When the conductor is finite at one end and infinitely long at the other, i.e., semi-infinite, then $\theta_1 = 90°$ and $\theta_2 = 0°$. In that case, intensity of the magnetic field at P,

$$B = \frac{\mu_0}{4\pi}\frac{I}{r}(\cos 90° + \cos 0°) = \frac{\mu_0}{4\pi}\frac{I}{r}$$

Numerical Examples:

X. *Find the value and direction of the Lorentz force acting on a particle with charge* $3.2 \times 10^{-19}C$, *moving at a velocity* $(3\hat{\imath} - 4\hat{\jmath})ms^{-1}$ *through a magnetic field* $\vec{B} = (2\hat{\imath} + 3\hat{\jmath})\ T$.

Answer:

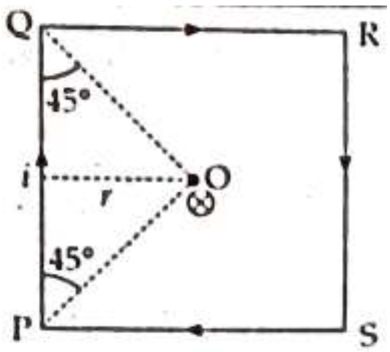
The Lorentz force on the charged particle

$$\vec{F} = q(\vec{v} \times \vec{B}) = q\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & -4 & 0 \\ 2 & 3 & 0 \end{vmatrix} = 3.2 \times 10^{-19} \times (9+8)\hat{k}\ N = 5.44 \times 10^{-18} N$$

in the z direction.

XI. *A conductor is bent in such a way that it makes a square frame of radius a. If a current i passes through the frame, what would be the intensity of the magnetic field at the center of the frame?*

Answer:



Following the figure, $PQRS$ is the square frame with the center at O. From the symmetry of the frame, the magnetic field at the center would be 4 times that generated by any side of the square. Let us isolate a side PQ. The field B_1 at O due to PQ is easy to calculate:

$$B_1 = \frac{\mu_0}{4\pi}\frac{i}{r}(\cos\theta_1 + \cos\theta_2) = \frac{\mu_0}{4\pi}\frac{i}{a/2}(\cos 45° + \cos 45°)$$

$$= \frac{\mu_0}{4\pi}\frac{2\sqrt{2}i}{a}\ \text{Tesla}$$

b. Magnetic field intensity at a point on the axis of a circular current carrying coil:

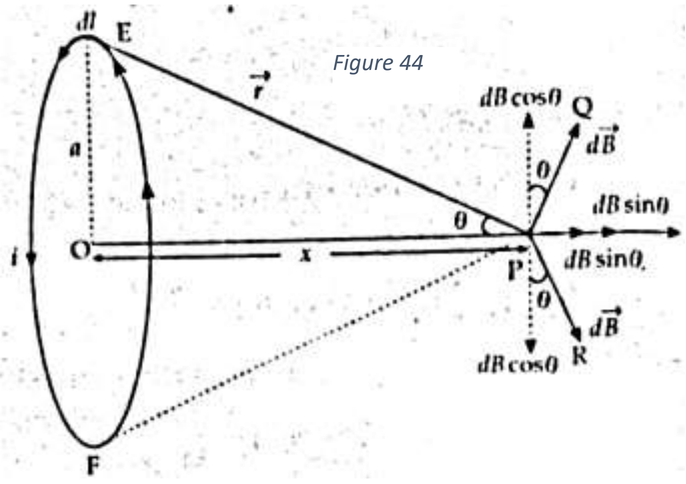


Figure 44

In the adjacent Figure 3, the circular conducting coil (with one loop) centered at O is carrying a steady current of i Amperes. A 3-D point of view, for ease of understanding is in Figure 4. The job is to calculate the magnetic field B at a point P on the axis of the loop at a distance x from O. Consider an infinitesimal length dl of the loop at point E, with the corresponding current element $id\vec{l}$. The displacement vector EP is $\vec{r}$. The direction of the magnetic field at P is along $\overrightarrow{PQ}$, perpendicular to the plane containing both $id\vec{l}$ and $\vec{r}$. Combining all these, the magnetic field at P due to $id\vec{l}$ is

$$d\vec{B} = \frac{\mu_0}{4\pi}\left(\frac{id\vec{l}\times\vec{r}}{r^3}\right)$$

$$\Rightarrow |d\vec{B}| = \frac{\mu_0}{4\pi}\frac{i\,dl\,r\sin 90°}{r^3} = \frac{\mu_0 i}{4\pi}\frac{dl}{r^2}$$

$$= \frac{\mu_0 i}{4\pi}\frac{dl}{(a^2+x^2)}$$

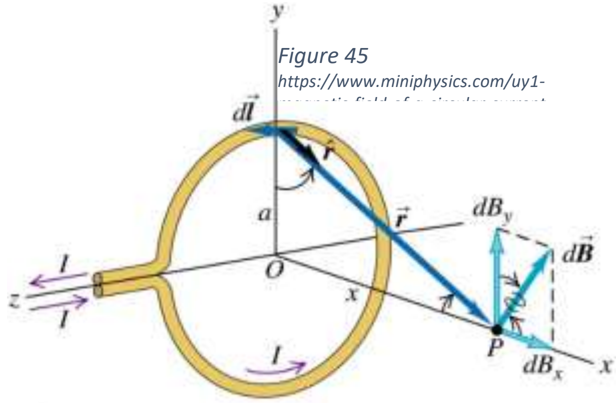


Figure 45
https://www.miniphysics.com/uy1-

Now, the components along the axis and perpendicular to the axis of the loop are, respectively, $dB\sin\theta$ and $dB\cos\theta$.

We can consider a similar current element of the loop $id\vec{l}$ at a point F exactly on the opposite side of it. Following the derivation above, the component of the magnetic field due to this element at P, perpendicular to the axis of the loop is $-dB\cos\theta$, exactly same and opposite to that due to the element at E. These two cancel each other out. Following the circular symmetry of the loop, we can see that for every current element on it, there is an equal and opposite element too, whose perpendicular components cancel each other, i.e., $\int dB\cos\theta = 0$. Thus the only remaining component of the force is along the axis:

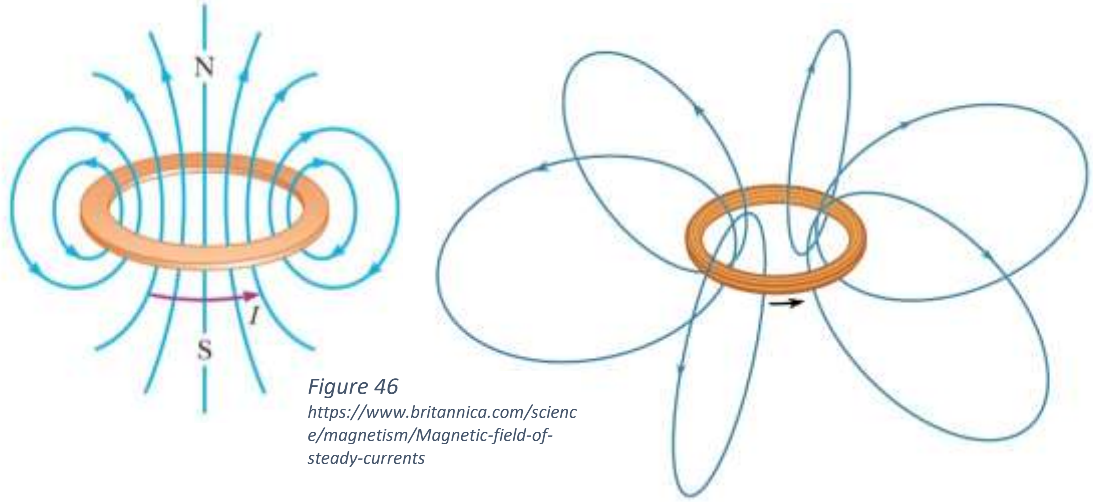


Figure 46
https://www.britannica.com/science/magnetism/Magnetic-field-of-steady-currents

$$B = \int dB\sin\theta = \int_0^{2\pi a}\frac{\mu_0 i}{4\pi}\frac{dl}{(a^2+x^2)}\sin\theta = \int_0^{2\pi a}\frac{\mu_0 i}{4\pi}\frac{dl}{(a^2+x^2)}\frac{a}{\sqrt{a^2+x^2}} = \frac{\mu_0 ia}{4\pi(a^2+x^2)^{\frac{3}{2}}}\int_0^{2\pi a} dl$$

$$= \frac{\mu_0 ia}{4\pi(a^2+x^2)^{\frac{3}{2}}}2\pi a$$

$$\Rightarrow \vec{B} = \frac{\mu_0 i}{2}\frac{a^2}{(a^2+x^2)^{\frac{3}{2}}}\hat{x}$$

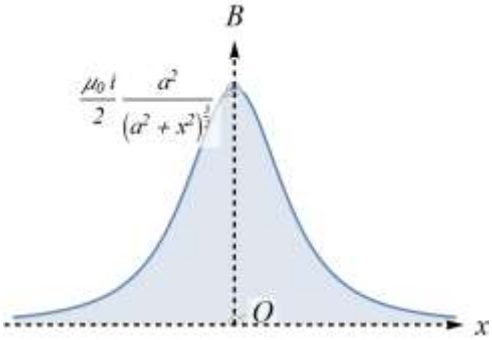
We can clearly see how this unidirectional magnetic field can (very roughly) mimic a real magnet with poles on both sides. Figure 5 tries to show this pictorially. Case 'C' below shows this in a mathematical way.

Special Cases:

A. We can imagine the coil to be comprised of n loops. Then the intensity increases to simply

$$\vec{B} = n\frac{\mu_0 i}{2}\frac{a^2}{(a^2+x^2)^{\frac{3}{2}}}\hat{x}$$

B. The field is maximum $\left(\frac{\mu_0 i}{2a}\right)$ at $x = 0$, i.e., at the center of the loop and symmetrically decreases on both sides of the loop with increasing x. The figure on the right plots the change of B with x.
Home Work: Calculate the double derivative of B w.r.t. x. At which point is it zero?



C. One fringe case is quite interesting. When P is at a large distance from the center of the loop, i.e., $x \gg a$,

$$\vec{B} = \frac{\mu_0 i}{2}\frac{a^2}{x^3}\hat{x} = \frac{\mu_0}{2\pi}\frac{i(\pi a^2)\hat{x}}{x^3} = \frac{\mu_0}{2\pi}\frac{i\vec{A}}{x^3}$$

where $\vec{A} = \pi a^2 \hat{x}$ is the area vector of the loop. Hence, $\vec{M} = i\vec{A}$ is the dipole moment of the current-carrying loop. That means $\vec{B} = \frac{\mu_0}{2\pi}\frac{\vec{M}}{x^3}$, which is the exact form of a bar magnet. Thus, at a distance, a circular loop just mimics a bar magnet.

c. Magnetic field intensity at a point on the axis of a current carrying solenoid:

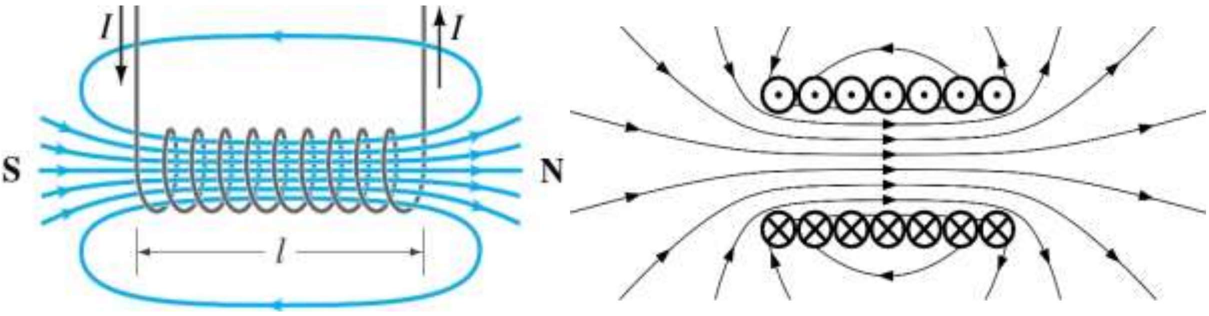


Figure 47 Solenoid (https://www.miniphysics.com/ss-magnetic-field-due-to-current-in-a-solenoid.html)

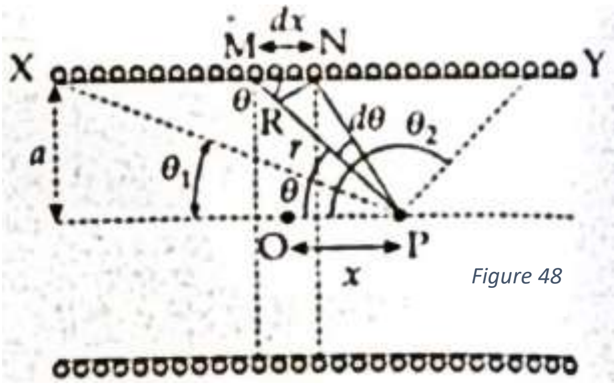


Figure 48

A solenoid whose length is much larger than its radius is called a long solenoid. Adjacent figures show the cross-section of the solenoid. Let's say that the radius of the solenoid$= a$, current through it is I, turn density (no. of loops per unit length)$= n$. To determine the magnetic field intensity of the solenoid at a point P on its axis, we consider a small length dx (MN) of the solenoid. The center of this length element O is at a distance x from the point P (along the axis). As dx is infinitesimally small, it is equivalent to a thin circular loop, two ends of which (M and N) are at angles θ and $\theta + d\theta$ with the axis at P, respectively.

Number of turns within $MN = n\,dx$. $MP = r = \sqrt{a^2 + x^2}$. Also, from Figure 7, $NR = r\,d\theta$ and $\frac{NR}{MN} = \sin\theta$.

$$\therefore NR = MN\sin\theta = dx\sin\theta \Rightarrow r\,d\theta = dx\sin\theta \Rightarrow dx = \frac{r\,d\theta}{\sin\theta}$$

$$\text{also, } \frac{a}{r} = \sin\theta$$

Hence, the magnetic field intensity at P due to dx length of the solenoid,

$$dB = \frac{\mu_0 I}{4\pi}\frac{2\pi(n\,dx)\,a^2}{(a^2+x^2)^{\frac{3}{2}}} = \frac{\mu_0 I}{4\pi}\left(\frac{2\pi n}{r}\right)\left(\frac{a}{r}\right)^2\frac{r}{\sin\theta}\,d\theta = \frac{\mu_0 I}{4\pi}(2\pi n)\sin^2\theta\frac{1}{\sin\theta}\,d\theta = \frac{\mu_0 nI}{2}\sin\theta\,d\theta$$

$$\Rightarrow B = \frac{\mu_0 nI}{2}\int_{\theta_1}^{\theta_2}\sin\theta\,d\theta = \frac{\mu_0 nI}{2}[-\cos\theta]_{\theta_1}^{\theta_2} = \frac{\mu_0 nI}{2}(\cos\theta_1 - \cos\theta_2)$$

Discussions:

A. From Figure 7 we can see that if the solenoid is very long and P is far from both ends of it, then $\theta_1 \approx 0°$ and $\theta_2 \approx 180°$. In that case, the magnetic field intensity becomes

$$B = \frac{\mu_0 nI}{2}(\cos 0° - \cos 180°) = \mu_0 nI$$

B. If P is on the left end of the (very long) solenoid, then $\theta_1 = 90°$ and $\theta_2 \approx 180°$. Then the magnetic field is $B = \frac{\mu_0 nI}{2}(\cos 90° - \cos 180°) = \frac{\mu_0 nI}{2}$

C. Similarly, for the right end, $\theta_1 \approx 0°$ and $\theta_2 = 90°$, and the field, $B = \frac{\mu_0 nI}{2}(\cos 0° - \cos 90°) = \frac{\mu_0 nI}{2}$

Thus, the field intensity at two ends of a solenoid is same and is half of that inside the solenoid.

Numerical Examples:

I. *A circular coil of radius 10 cm has 100 turns in it. What is the intensity of the magnetic field at its center if the current through the coil is 5 A?*

Answer:

As $x = 0$, magnetic field-intensity at the center of the coil,

$$B = \frac{\mu_0 nI}{2a}$$

Here $n = 100, a = 10\ cm = 0.1\ m, I = 5\ A$, and $\mu = 4\pi \times 10^{-7} Hm^{-1}$.

$$\therefore B = \frac{4\pi \times 10^{-7} \times 100 \times 5}{2 \times 0.1}\ T = \pi \times 10^{-3}\ T$$

II. *A solenoid is made by wrapping densely-packed thin wire around a non-conducting cylinder. The length of the solenoid is twice its radius. What is the magnetic field at its center if current passing through the coil is I?*

Answer:

As $L = 2a$, following the adjacent figure, $\cos\theta_1 = \frac{\frac{L}{2}}{\sqrt{2}a} = \frac{1}{\sqrt{2}}$, and $\cos(\pi - \theta_2) = \frac{\frac{L}{2}}{\sqrt{2}a} = \frac{1}{\sqrt{2}} \Rightarrow \cos\theta_2 = -\frac{1}{\sqrt{2}}$. If the total number of turns in the solenoid is N, then the magnetic field at the center,

$$B = \frac{\mu_0\left(\frac{N}{L}\right)I}{2}(\cos\theta_1 - \cos\theta_2) = \frac{\mu_0 NI}{\sqrt{2}L}$$

III. *What is the magnetic field intensity at the center of a semi-circular wire-frame of length L and carrying current I?*

Answer:

If the radius of the semi-circle is r, then $L = \pi r \Rightarrow r = L/\pi$. Hence the field intensity at the center

$$B = \frac{1}{2}\frac{\mu_0 I}{2r} = \frac{\mu_0 \pi I}{4L}$$