(*ii*) Verify Caley-Hamilton's theorem for the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 2 & 3 & 2 \end{pmatrix}$$

Hence compute A^{-1}. 3 + 2

(*b*) (*i*) Find all real λ for which the rank of the matrix A in 2, where

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & \lambda \\ 5 & 7 & 1 & \lambda^2 \end{pmatrix}.$$ 3

(*ii*) If $X_1, X_2, \ldots, X_r$ be r eigen vectors of an $n \times n$ matrix A corresponding to r distinct eigen values $\lambda_1, \lambda_2 \ldots, \lambda_r$ respectively, then prove that $X_1, X_2, \ldots, X_r$ are linearly independent. 5

(*iii*) λ is an eigen value of a real skew symmetric matrix. Prove that

$$\left|\frac{1-\lambda}{1+\lambda}\right| = 1.$$ 2

2017

MATHEMATICS

[Honours]

(CBCS)

[First Semester]

PAPER – C2T

Full Marks : 60

Time : 3 hours

The figures in the right hand margin indicate marks

UNIT – I

(*Classical Algebra*)

1. Answer any *one* question : 2 × 1

(*a*) If $x + iy$ moves on the straight line $3x + 4y + 5 = 0$, then find the minimum value of $|x + iy|$. 2

(*b*) Solve the equation $x^5 + x^4 + x^3 + x^2 + x + 1 = 0$. 2

2. Answer any *two* questions : 5 × 2

(*a*) If $(1 + i\tan\alpha)^{1 + i\tan\beta}$ can have real values, then show that one of them is $(\sec\alpha)^{\sec^2\beta}$. 5

(*b*) Show that the condition that the sum of two roots of the equation $x^4 + mx^2 + nx + p = 0$ be equal to the product of the other two roots is $(2p - n)^2 = (p - n)(p + m - n)^2$. 5

(*c*) If $a_1, a_2, \ldots, a_n$ be n real positive quantities then prove that

$$A.M. \geq G.M. \geq H.M.$$ 5

3. Answer any *one* question : 10 × 1

(*a*) (*i*) If $x + \frac{1}{x} = 2\cos\alpha$, $y + \frac{1}{y} = 2\cos\beta$, $z + \frac{1}{z} = 2\cos\gamma$, and $x + y + z = 0$ then prove that

$$\sum \sin 4\alpha = 2\sum \sin(\beta + \gamma)$$

and $\sum \cos 4\alpha = 2\sum \cos(\beta + \gamma)$ 5

(*ii*) If the equation whose roots are squares of the roots of the cubic $x^3 - ax^2 + bx - 1 = 0$ is identical with this cubic, prove that either $a = b = 0$ or $a = b = 3$ or $a_1 b$ are the roots of the equation $t^2 + t + 2 = 0$. 5

(*b*) (*i*) If a, b, c, x, y, z be all real numbers and $a^2 + b^2 + c^2 = 1$, $x^2 + y^2 + z^2 = 1$ then prove that $-1 \leq ax + by + cz \leq 1$.

If $a_1, a_2, \cdots a_n$ be n positive rational numbers and $s = a_1 + a_2 + \cdots + a_n$, prove that

$$\left(\frac{s}{a_1} - 1\right)^{a_1} \left(\frac{s}{a_2} - 1\right)^{a_2} \cdots \left(\frac{s}{a_n} - 1\right)^{a_n} \leq (n-1)^s.$$ 2 + 3

(*ii*) If the equation $x^3 + px^2 + qx + r = 0$ has a root $\alpha + i\alpha$ where p, q, r and α are real, prove that $(p^2 - 2q)(q^2 - 2pr) = r^2$.

Hence solve the equation

$$x^3 - x^2 - 4x + 24 = 0.$$ 3 + 2

UNIT – II

(*Sets and Integers*)

4. Answer any *five* questions : 2 × 5

(*a*) Prove that intersection of two equivalence relations is also an equivalence relation. 2

(*b*) Prove that square of any integer is of the form $3k$ or $3k+1$. 2

(*c*) Examine if the relation ρ on the set $\mathbb{Z}$ is an equivalence relation or not

$$\rho = \{(a,b) \in \mathbb{Z}\times\mathbb{Z} : |a-b| \leq 3\}.$$ 2

(*d*) Prove that, there exists no integer in between 0 and 1. 2

(*e*) Let $P = \{n \in \mathbb{Z} : 0 \leq n \leq 5\}, Q = \{n \in \mathbb{Z} : -5 \leq n \leq 0\}$ be two sets. Prove that cardinality of two sets are equal. 2

(*f*) If $s_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ then prove that

$$s_n > \frac{2n}{n+1}$$

if $n \geq 1$. 2

(*g*) If X and Y are two non-empty sets and $f : X \rightarrow Y$ be an onto mapping, then for any subsets A and B of Y, prove that

$$f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B).$$ 2

(*h*) (*i*) State the Fundamental theorem of Arithmetic.

(*ii*) If a divides b, then prove that every divisor of a divides b. 2

5. Answer any *one* question : 5 × 1

(*a*) (*i*) Prove that $1^n - 3^n - 6^n + 8^n$ is divisible by 10 $\forall n \in \mathbb{N}$. 2

(*ii*) Find integers u and v satisfying $20u + 63v = 1$. 3

(*b*) (*i*) State the division algorithm on the set of integers. 1

(*ii*) Find integers s and t such that

$$gcd\,(341, 1643) = 341s + 1643t.$$ 2

(*iii*) Using the theory of congruence for finding the remainder when the sum $1^5 + 2^5 + 3^5 + \cdots + 100^5$ is divided by 5. 2

UNIT – III

(*System of Linear Equations*)

6. Answer any *two* questions : 2 × 2

(*a*) Solve the system of equations :

$$x + 2y - z - 3w = 1$$
$$2x + 4y + 3z + w = 3$$
$$3x + 6y + 4z - 2w = 5$$

if possible. 2

(*b*) For what values of k the system of equations

$$x + 2y + 3z = kx$$
$$2x + y + 3z = ky$$
$$2x + 3y + z = kz$$

has a non-trivial solution. 2

(*c*) Determine k so that the set $\{(1, 2, 1), (k, 3, 1), (2, k, 0)\}$ is linearly dependent. 2

7. Answer any *one* question : 5 × 1

(*a*) Determine the conditions for which the system

$$x + y + z = 1$$
$$x + 2y - z = b$$
$$5x + 7y + az = b^2$$

admits of (*i*) only one solution.

(*ii*) no solution.

(*iii*) many solutions. 5

(*b*) (*i*) Obtain the fully row reduced normal form of the matrix : 2

$$\begin{pmatrix} 0 & 0 & 1 & 2 & 1 \\ 1 & 3 & 1 & 0 & 3 \\ 2 & 6 & 4 & 2 & 8 \\ 3 & 9 & 4 & 2 & 10 \end{pmatrix}.$$

(*ii*) For what values of k, the planes $x - 4y + 5z = k$, $x - y + 2z = 3$, $2x + y + z = 0$ intersect in a line. 3

UNIT – IV

(*Linear Transformation and Eigenvalues*)

8. Answer any *two* questions : 2 × 2

(*a*) Find the rank of the matrix :

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix}$$

if two straight lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are coincident. 2

(*b*) Show that the rank of a skew symmetric matrix cannot be 1. 2

(*c*) State Cayley-Hamilton theorem and using theorem find A^{-1}, where

$$A = \begin{pmatrix} 2 & 1 \\ 3 & 5 \end{pmatrix}.$$ 2

9. Answer any *one* question : 10 × 1

(*a*) (*i*) If $A = \begin{pmatrix} \frac{1}{v_2} & -\frac{1}{v_2} \\ \frac{1}{v_2} & \frac{1}{v_2} \end{pmatrix}$, $X = (x_1, x_2)^T$ and $Y = (y_1, y_2)^T$. Verify by means of the transformation $X = AY$ that $x_1^2 + x_2^2$ is transformed to $y_1^2 + y_2^2$.

Find the dimension of the subspace $\mathbb{R}^3$ defined by

$$S = \{(x, y, z) \in \mathbb{R}^3 : x + 2y = z, 2x + 3z = y\}.$$

3 + 2