Computer Graphics
Curves (Bezier and B-Spline)

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Relevance and importance of Computer Graphics: The relevance and importance of Computer Graphics lies in its applications. In engineering application, the ability to quickly visualize newly designed shapes is indispensable. Interactive Computer Graphics allows the physician to interpret this large volume of data in new and useful ways. 3D Computer Graphics has many applications in the entertainment, industry, most importantly video games and visual effects in movie production. In medicine, Computer Graphics is used to visualize data obtained from medical imaging, such as CT scans. Computer-aided design relies on Computer Graphics for modelling and visualising products, buildings, cars etc. The same modelling techniques are used for creating 3D-printed objects.

Introduction:

Today there are very few aspects of our lives not affected by Computers. In most of the cases in our daily life the applications of Computer Graphics occur. Computer Graphics is the display, storage and manipulation of images and data for the visual representation of a system. Computer Graphics is a part of drawing pictures, lines, charts etc., using computers with the help of programming. Computer Graphics is made up of number of pixels. Some applications of Computer Graphics include Computer aided design (CAD), presentation graphics, entertainment (animation, games), education and training, image processing, computer art, scientific visualization, graphical user interfaces etc.

Curves

Modelling of objects is the major concept of Computer Graphics. So, in this regard, we want to represent the objects in terms of their geometrical property, in size, shape etc. and how they are interrelated. In Computer Graphics applications, we need surfaces and some smooth curves for modelling shapes, most real-world objects are inherently smooth. Generally, Bezier Curves are widely used in graphics to represent smooth curves. Quadratic and Cubic Bezier curves are most in common to represent different curves. Higher degree curves are more computationally expensive to evaluate. When more complex shapes are needed, low order Bezier Curves are patched together, to produce a Composite Bezier Curve.

Linear Bezier Curve (Line Segment):

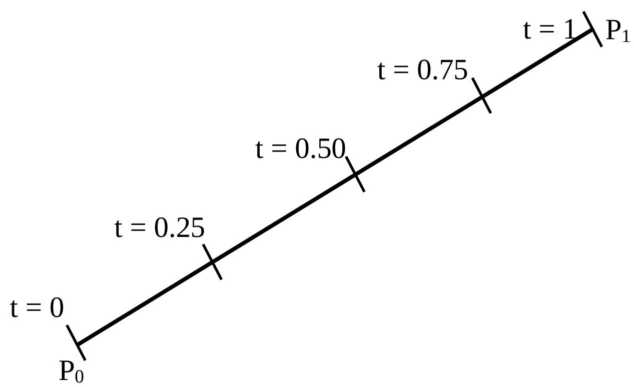
Let us consider two control points P_0 and P_1. Now the parametric equation of the line segment joining between P_0 and P_1 is

$$P(t) = (1-t)\, P_0 + tP_1 \text{ where } 0 \le t \le 1$$

At, t = 0, $P(0)=P_0$ and at t = 1, $P(1) = P_1$

Similarly, at t = 0.5, $P(0.5) = \frac{1}{2}(P_0 + P_1)$

So, $P(0.25) = \frac{3P_0+P_1}{4}$ and $P(0.75) = \frac{P_0+3P_1}{4}$



If $P_0 = (x_0 , y_0)$ and $P_1 = (x_1 , y_1)$ and $P(t) = [x(t), y(t)]$

Then, $x(t) = (1-t)\, x_0 + tx_1$

$y(t) = (1-t)y_0 + ty_1$

Some Observations:

i) Co-efficient of P_0 and P_1 are (1-t) and t, and their sum is equal to one. So, $P(\lambda) = \lambda P_1 + (1-\lambda)P_2$, $\lambda \in [0,1]$

ii) For fixed t, say t*, $P(t^*) = (1-t^*)\, P_0 + t^*P_1$. So, for fixed value of t, say t*, will represent a fixed point $[P(t^*)]$ on the line segment joining between P_0 and P_1

$P(t) = (1-t)\, P_0 + tP_1$, $0 \le t \le 1$

So, this is the parametric equation of the line segment from P_0 to P_1.

Cubic Bezier Curve:

Problem: Obtain a cubic polynomial curve with four Control Points b_0, b_1, b_2, b_3

Ans: The line segment joining b_0 and b_1 is,

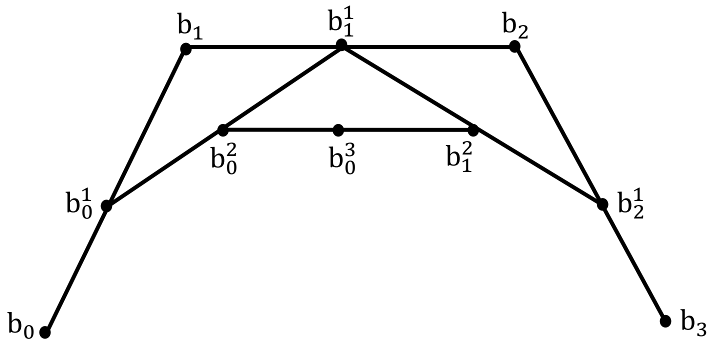
b_0^1 (t) = (1-t)b_0 + tb_1; $0 \le t \le 1$ .... (1)

Line segment joining b_1 and b_2 is,

b_1^1 (t) = (1-t)b_1 + tb_2; $0 \le t \le 1$ .... (2)

Line segment joining b_2 and b_3 is,

b_2^1 (t) = (1-t)b_2 + tb_3; $0 \le t \le 1$ .... (3)



Line segment joining b_0^1 (t) and b_1^1 (t) is,

b_0^2 (t) = (1-t) b_0^1 (t) + t b_1^1 (t); $0 \le t \le 1$ .... (4)

Line segment joining b_1^1 (t) and b_2^1 (t) is,

b_1^2 (t) = (1-t) b_1^1 (t) + t b_2^1 (t); $0 \le t \le 1$ .... (5)

Line segment joining b_0^2 (t) and b_1^2 (t) is,

b_0^3 (t) = (1-t) b_0^2 (t) + t b_1^2 (t); $0 \le t \le 1$ .... (6)

From (6), using (4) and (5) we get,

$b_0^3(t) = (1-t)[(1-t)b_0^1(t) + tb_1^1(t)] + t[(1-t)b_1^1(t) + tb_2^1(t)]$

$= (1-t)^2 b_0^1(t) + 2(1-t)tb_1^1(t) + t^2 b_2^1(t)$

$= (1-t)^2\{(1-t)b_0 + tb_1\} + 2t(1-t)\{(1-t)b_1 + tb_2\} + t^2\{(1-t)b_2 + tb_3\}$

$= (1-t)^3 b_0 + 3(1-t)^2 tb_1 + 3(1-t)t^2 b_2 + t^3 b_3$, where $0 \le t \le 1$ .... (7)

So, we can write C (t) as,

C (t) = $\sum_{i=0}^{3} B_i^3 b_i$ .... (8)

Where, $B_i^3 = \binom{3}{i}(1-t)^{3-i} t^i$

So, C (t) = TMG is matrix form,

Where, T = $(t^3 \;\; t^2 \;\; t \;\; 1)_{(1 \times 4)}$

$$M = \begin{pmatrix} 1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}_{(4 \times 4)} \quad \text{and} \quad G = \begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{pmatrix}_{(4 \times 1)}$$

$$\therefore c(t) = \begin{pmatrix} t^3 & t^2 & t & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{pmatrix} \quad \text{.........................} \quad (9)$$

$C(t) = (1 - t^3 - 3t + 3t^2)b_0 + 3(t - 2t^2 + t^3)b_1 + 3(t^2 - t^3)b_2 + t^3 b_3$

This is called Bezier Curve or Cubic polynomial Curve. So, the resulting curve b_0^3 (t) [c(t)] is the cubic Bezier Curve defined by four control points. For all Bezier Curves, we keep t in the range [0,1]. So, if we consider a sequence of points b_0, b_1, b_2, ... b_n, we can directly evaluate the co-efficient of each point. For a class of curves known as Bezier Curves, the Co-efficient are defined by the Bernstein polynomials:

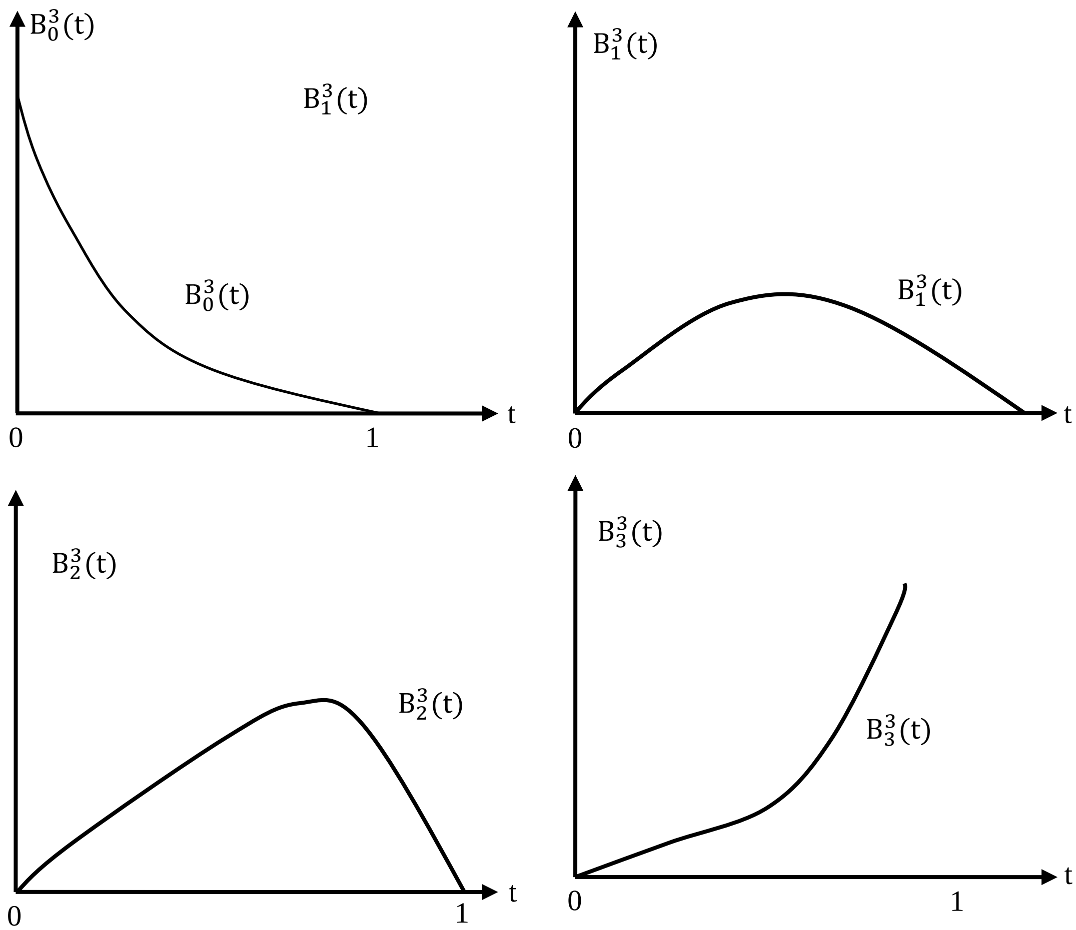
$$b_0^n(t) = \sum_{i=0}^{n} \binom{n}{i} (1-t)^{n-i} . t^i . b_i \quad \text{...} \quad (10)$$

$$= \sum_{i=0}^{n} B_i^n(t) . b_i$$

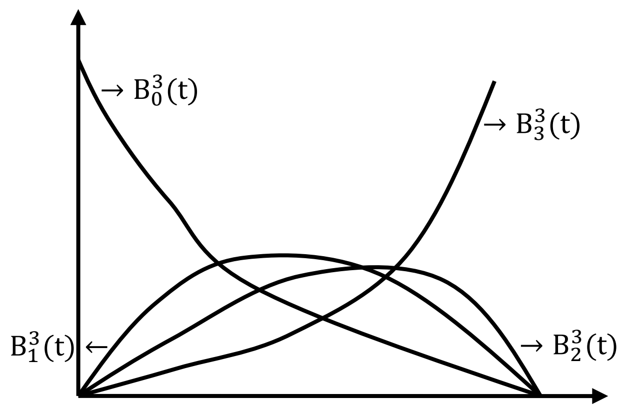
Where, $B_i^n(t) = \binom{n}{i} (1-t)^{n-i} . t^i$ .... (11)

are called the Bernstein basis functions / polynomials. For example, Cubic Bezier Curves have the following Co-efficient:

$$\left. \begin{aligned} B_0^3(t) &= (1-t)^3 \\ B_1^3(t) &= 3(1-t)^2 t \\ B_2^3(t) &= 3(1-t)t^2 \\ B_3^3(t) &= t^3 \end{aligned} \right\} \quad \text{.......................................} \quad (12)$$



The four Bezier blending functions for cubic curves (n=3). Now in one frame above four blending functions are as follows:



Four Bezier blending functions for cubic curves
(n=3)

Properties of Bezier Curve, C (t):

i) At t = 0, C (0) = b_0 (starting point or initial point of the curve).

ii) At t = 1, C (1) = b_3 (Terminating point of the curve).

iii) Convexity of the basis/blending functions:

For all values of t ∈ [0,1], the sum of blending functions is 1.

$$\sum_{i=0}^{n} B_i^n(t) = 1 \text{ .... (13)}$$

For the cubic curve, this can be shown that,

$$\{(1-t)+t\}^3 = (1-t)^3 + 3(1-t)^2 t + 3(1-t)t^2 + t^3 = 1 \text{ .... (14)}$$

In general case, we have

$$\{(1-t)+t\}^n = \sum_{i=0}^{n} \binom{n}{i} (1-t)^{n-i} t^i = 1 \text{ .... (15)}$$

Also, it can be easily shown that the blending functions are always non-negative:

$$B_i^n(t) \geq 0$$

iv) Convex Hull Property:

Since $B_i^n(t) \geq 0$, C (t) is a convex combination of the control points. Thus, Bezier Curves always lie within the convex hull of the control points.

v) Linear Precision:

When the control points lie on a straight line, then the corresponding Bezier curve will also be a straight line.

vi) $\frac{d}{dt} = \{c(t)\} = \frac{d}{dt}\{(1-t)^3 b_0 + 3(1-t)^2 t b_1 + 3(1-t)t^2 b_2 + t^3 b_3\}$

$= -3(1-t)^2 b_0 + 3(1-t)^2 b_1 - 6(1-t)tb_1 + 6t(1-t)b_2 - 3t^2 b_2 + 3t^2 b_3$

$= 3[(1-t)^2(b_1 - b_0) + 2(1-t)t(b_2 - b_1) + t^2(b_3 - b_2)]$

=

$$3\sum_{i=0}^{2} B_i^2 . \Delta b_i$$

, where $B_i^2 = \binom{2}{i}(1-t)^{2-i}\, t^i$ and $\Delta b_i = b_{i-1} - b_i$

$\frac{d}{dt}\{C(t)\}_{t=0}$ =3(b_1-b_0)=Tangent at the initial point (b_0)

and $\frac{d}{dt}\{C(t)\}_{t=1}$=3($b_3$-$b_2$)=Tangent at the terminating point.

$\frac{d^2}{dt^2}[c(t)] = 6[(1-t)(b_2 - 2b_1 + b_0) + t(b_3 - 2b_2 + b_1)]$

$= 6\sum_{i=0}^{1} B_i^1 \Delta^2 b_i$, where $B_i^1 = \binom{1}{i}(1-t)^{1-i} t^i$

∴ Curvature at the initial point (at t = 0)

= 6(b_2-2b_2+b_0)

and curvature at the terminal point (at t = 1)

= 6 (b_2-2b_2+b_1)

Also, we note that, $\sum_{i=0}^{3} B_i^3 = 1$

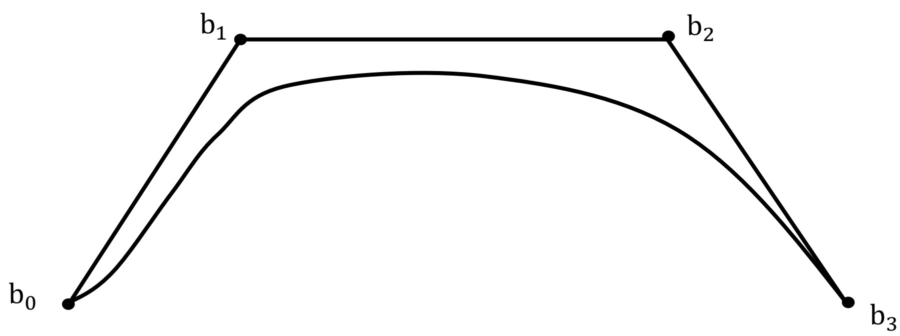
$(1-t)^3 + 3(1-t)^2 t + 3(1-t)t^2 + t^3 = 1$

This proves that, the curve is within the quadrant, corner points are b_0, b_1, b_2, b_3.

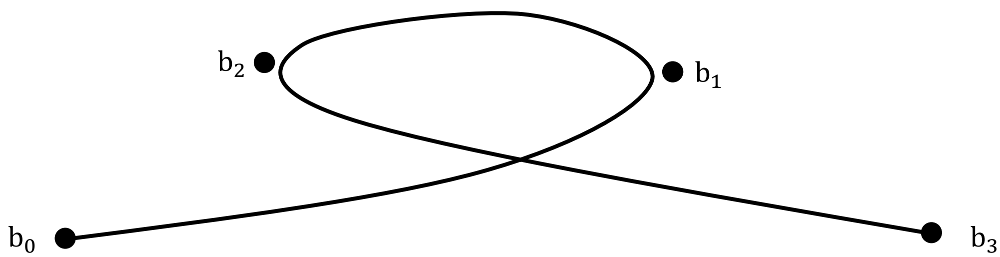
vii) Nature of the curve depends on the order of the control points:

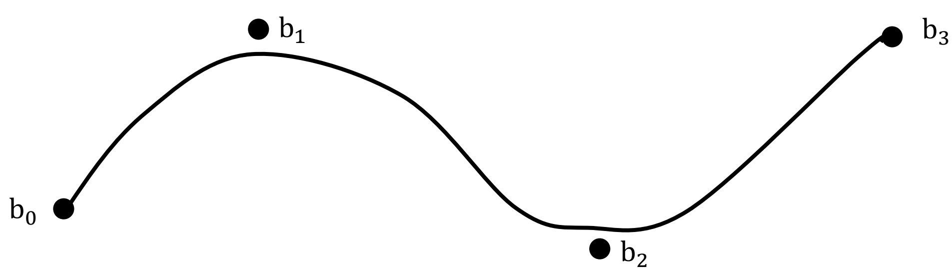
Let us consider four control points b_0, b_1, b_2, b_3. So, the corresponding Bezier Curve will be,

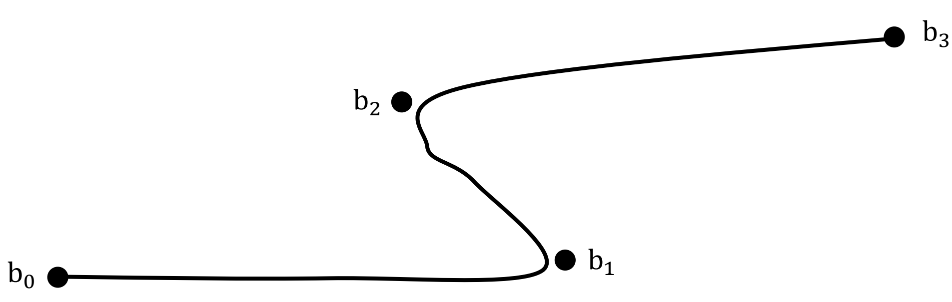
C (t) = $(1-t)^3 b_0 + 3(1-t)^2 t b_1 + 3(1-t)t^2 b_2 + t^3 b_3$, where $0 \le t \le 1$



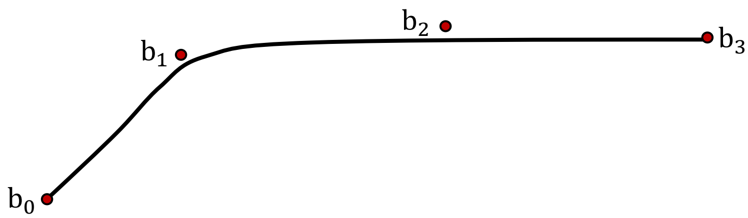
Now, we see that the nature of the Bezier Curve on the control points on which order they are situated in placed.





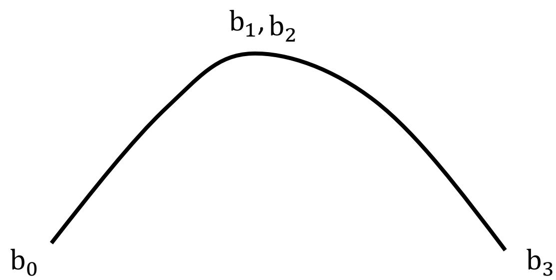


viii) If three points are collinear, then the curve is closer to the straight line.



If four points are collinear, then Bezier Curve will be straight line itself.

If two control points are coincident then,



B-Spline Curve:

The term spline comes from engineering drawing, where a spline is a piece of flexible wood to draw smooth curves. A spline is a parametric curve defined by control points. The control points are adjusted by the user to control the shape of the curve. Two features of the Bezier or Bernstein basis function limit the flexibility of the resulting curves. Firstly, the number of control points defines the order of the resultant curve. Secondly, any point on the curve is a result of blending the values of all defining vertices, and hence any change of one vertex is felt throughout the entire curve. B-spline curves overcome these two important limitations of Bezier curves. Here (i) the order or degree of the curve is independent of the number of control points, and (ii) the control points have only local control over the shape of the spline or surface.

Here we consider (n+1) ordered control points Q_0, Q_1, Q_2, Q_3, ..., Q_n. Let i-th segment of the curve be

$\text{þ}_i(t) = x_0(t)Q_{i-1} + x_1(t)Q_i + x_2(t)Q_{i+1} + x_3(t)Q_{i+2}$ .... (1)

Where, $0 \leq t \leq 1$ and $1 \leq i \leq n-2$

Number of segments are (n-2). Here, $x_0(t)$, $x_1(t)$, $x_2(t)$ are $x_3(t)$ are polynomial in t of degree three.

Next segment is given by,

$\text{þ}_{i+1}(t) = x_0(t)Q_i + x_1(t)Q_{i+1} + x_2(t)Q_{i+2} + x_3(t)Q_{i+3}$ .... (2)

for $0 \leq t \leq 1, 0 \leq i \leq n-3$

Now, positional continuity of $\text{þ}_i(t)$ and $\text{þ}_{i+1}(t)$ implies

$[\text{þ}_i(t)]_{t=1} = [\text{þ}_{i+1}(t)]_{t=0}$

$x_0(1)Q_{i-1} + x_1(1)Q_i + x_2(1)Q_{i+1} + x_3(1)Q_{i+2}$

$= x_0(0)Q_i + x_1(0)Q_{i+1} + x_2(0)Q_{i+2} + x_3(0)Q_{i+3}$

So, $x_0(1) = 0$ and $x_3(0) = 0$ [Since there is no Q_{i-1} is R.H.S. and no Q_{i+3} is L.H.S] and

$x_1(1) = x_0(0)$, $x_2(1) = x_1(0)$, $x_3(1) = x_2(0)$ .... (3)

We get, these above relations by comparing co-efficient of Q_i, Q_{i+1}, Q_{i+2}

So, we get, these five conditions above.

Now, from tangential continuity we get,

$$[p_i'(t)]_{t=1} = [p_{i+1}'(t)]_{t=0}$$

This gives, $x_0'(1) = 0$ and $x_3'(0) = 0$ and

$x_1'(1) = x_0'(0), x_2'(1) = x_1'(0), x_3'(1) = x_2'(0)$.................................. (4)

These also gives, five conditions.

From curvature continuity, we know 2nd order derivative are also equal.

$x_1''(1) = 0, x_3''(0) = 0$

$x_1''(1) = x_0''(0), x_2''(1) = x_1''(0), x_3''(1) = x_2''(0)$ .... (5)

These are also five conditions

Also, one additional condition is that, curve remains invariant if and only it,

$x_0(t) + x_1(t) + x_2(t) + x_3(t) = 1$ .... (6)

So, theoretically this problem can be solved because, there are 16 unknowns and 16 conditions (5+5+5+1),

So, $x_0(t) + x_1(t) + x_2(t) + x_3(t) = 1$

$x_0(t) = x_3(1-t)$

$x_1(t) = x_2(1-t)$ .... (7)

Let, $x_2(t) = a_{20}t^3 + a_{21}t^2 + a_{22}t + a_{23}$

And $x_3(t) = a_{30}t^3 + a_{31}t^2 + a_{32}t + a_{33}$, $0 \leq t \leq 1$

Now, from (3), $x_3(0) = 0$. Putting this value in $x_3(t)$ we get,

$a_{33} = 0$

Also, $x_3'(0) = 0 \Rightarrow a_{32} = 0$ and $x_3''(0) = 0 \Rightarrow a_{31} = 0$

$\therefore$ $x_3(t) = a_{30}t^3$ .... (8)

$x_3'(t) = 3a_{30}t^2$ and $x_3''(t) = 6a_{30}t, x_2(0) = x_3(1)$

$\Rightarrow a_{23} = a_{30}$ [using (7) and (8)]

Now, $x_2'(0) = x_3'(1) \Rightarrow a_{22} = 3a_{30}$

And, $x_2''(0) = x_3''(1) \Rightarrow 2a_{21} = 6a_{30}$ [using (7) and (8)]

$\Longrightarrow a_{21} = 3a_{30}$

So, we get, $x_3(t) = a_{30}t^3$ and $x_2(t)= a_{20}t^3 + a_{30}(3t^2+3t + t)$ .... (9)

From, $x_0(0) + x_1(0) + x_2(0) + x_3(0) = 1$ [$\because$ (6)holds for all t is o to 1]

$\Rightarrow x_3(1) + x_2(1) + x_2(0) = 1$ [Since $x_0(0) = x_3(1)$, $x_1(0) = x_2(1)$ from (7) and $x_3(0)=0$ from (8)]

$\therefore a_{30} + a_{20} + 7a_{30} + a_{30} = 1$ [using (8) and (9)]

$\Rightarrow a_{20} + 9a_{30} = 1$ .... (10)

From (6) we get, $x_0'(t) + x_1'(t) + x_2'(t) + x_3'(t) = 0$

For t = 0, $x_0'(0) + x_1'(0) + x_2'(0) + x_3'(0) = 0$

$\Rightarrow -X_3'(1) - X_2'(1) + X_2'(0) = 0$ [using (7) and since $x_3'(0) = 0$]

$-3a_{30} - 3a_{20} - 9a_{30} + 3a_{30} = 0$

$\Rightarrow a_{20} + 3a_{30} = 0$ .... (11)

From (10) and (11) we get, $a_{30} = \frac{1}{6}$ and $a_{20} = -\frac{1}{2}$

$\therefore x_0(t) = \frac{1}{6}(1-t)^3 = -\frac{1}{6}t^3 + \frac{1}{2}t^2 - \frac{1}{2}t + \frac{1}{6}$

$x_1(t) = \frac{1}{2}t^3 - t^2 + 0.t + \frac{2}{3}$

$x_2(t) = -\frac{1}{2}t^3 + \frac{1}{2}t^2 + \frac{1}{2}t + \frac{1}{6}$

$x_3(t) = \frac{1}{6}t^3$

Here, the i – th segment of the B-spline curve can be written as,

$P_i(t) = x_0(t)Q_{i-1} + x_1(t)Q_i + x_2(t)Q_{i+1} + x_3(t)Q_{i+2}$

Where, $0 \leq t \leq 1$ and i = 1,2, ..., n-2.

Then we get,

$x_0(t) = -\frac{1}{6}t^3 + \frac{1}{2}t^2 - \frac{1}{2}t + \frac{1}{6}$

$x_1(t) = \frac{1}{2}t^3 - t^2 + 0.t + \frac{2}{3}$

$x_2(t) = -\frac{1}{2}t^3 + \frac{1}{2}t^2 + \frac{1}{2}t + \frac{1}{6}$

$x_3(t) = \frac{1}{6}t^3$

The curve remains invariant if sum of these $x_i(t)$ is equal to one (1).

Matrix notation of B-spline Curve:

In matrix form, $p_i(t)$ can be written as,

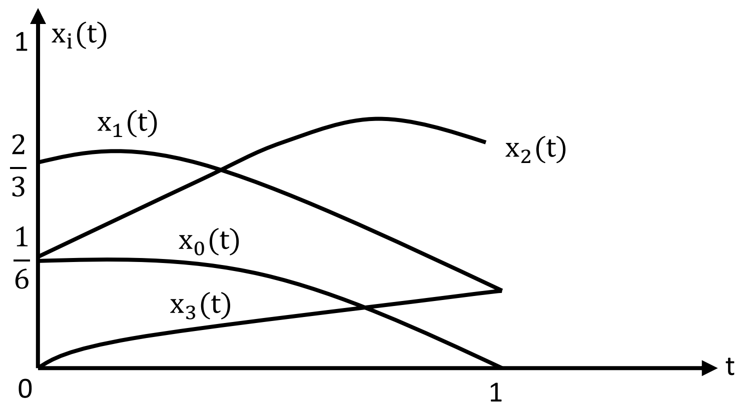
$p_i(t) = T.M_{BS}.G_{BS}$, where $T = (t^3 \quad t^2 \quad t \quad 1)$

$$M_{BS} = \begin{pmatrix} -\frac{1}{6} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{6} \\ \frac{1}{2} & -1 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 6 & \frac{2}{3} & -\frac{1}{6} & 0 \end{pmatrix}_{(4\times4)} \quad \text{and} \quad G_{BS} \begin{pmatrix} Q_{i-1} \\ Q_i \\ Q_{i+1} \\ Q_{i+2} \end{pmatrix}_{(4\times1)}$$

Let, $T. M_{BS} = B_{BS}$, then $p_i(t) = B_{BS}. G_{BS}$

Where, $B_{BS} = [x_0(t), x_1(t), x_2(t), x_3(t)]$

Geometrical representation of B_{BS} (Blending functions of B-spline curve):



For a particular value of t, values of the blending functions will be obtain from the ordinate at t = t_1

Some derivations of B-spline Curve:

1) Starting point of $p_i(t) = [p_i(t)]_{t=0}$

$$= \frac{1}{6}Q_{i-1} + \frac{2}{3}Q_i + \frac{1}{6}Q_{i+1}$$

2) Terminating point of $p_i(t) = [p_i(t)]_{t=1}$

$$= \frac{1}{6}Q_i + \frac{2}{3}Q_{i+1} + \frac{1}{6}Q_{i+2}$$

3) Tangent at the starting point and terminating point:

$x_0'(t) = -\frac{1}{2}(1-t)^2$	$x_0'(0) = -\frac{1}{2}$	$x_0'(1) = 0$
$x_1'(t) = \frac{3}{2}t^2 - 2t$	$x_1'(0) = 0$	$x_1'(1) = -\frac{1}{2}$
$x_2'(t) = -\frac{3}{2}t^2 + t + \frac{1}{2}$	$x_2'(0) = \frac{1}{2}$	$x_2'(1) = 0$
$x_3'(t) = \frac{1}{2}t^2$	$x_3'(0) = 0$	$x_3'(1) = \frac{1}{2}$

∴ Tangent at the initial point = $[p_i'(t)]_{t=0} = -\frac{1}{2}Q_{i-1} + \frac{1}{2}Q_{i+1} = \frac{1}{2}(Q_{i+1} - Q_{i-1})$

Tangent at the terminating point = $[p_i'(t)]_{t=1} = \frac{1}{2}(Q_{i+2} - Q_i)$

4) Similarly, Curvature of B-spline curve (i-th segment, $p_i(t)$) is given by,

At the starting point, $\left[p_i''(t)\right]_{t=0}$=$Q_{i-1} - 2Q_i + Q_{i+1}$

At the terminating point, $\left[p_i''(t)\right]_{t=1}$=$Q_i - 2Q_{i+1} + Q_{i+2}$

Since, $x_0''(t) = (1-t)$ $\quad$ $x_2''(t) = (1-3t)$

$x_1''(t) = (3t-2)$ $\quad$ $x_3''(t) = t$

Relate the B-spline curve with Bezier Curve:

In other words, it can be said that how the control points of Bezier curve are related with control points of the B-spline curve. Let, both curves are equal to C(T)

∴ $C(T) = T\ M_{BZ}\ G_{BZ} = T\ M_{BS}\ G_{BS}$

or, $M_{BZ}\ G_{BZ} = M_{BS}\ G_{BS}$ (operating both sides by T^{-1})

or, $M_{BZ}^{-1}\ M_{BZ}\ G_{BZ} = M_{BZ}^{-1}\ M_{Bs}\ G_{Bs}$ $\quad$ Here, $G_{BZ} = \begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{pmatrix}$

or, $G_{BZ} = M_{BZ}^{-1}\ M_{BS}\ G_{BS}$

We know that, $M_{BZ} = \begin{pmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

$$M_{BZ}^{-1} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{3} & 1 \\ 0 & \frac{1}{3} & \frac{2}{3} & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

Let, $\begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{3} & 1 \\ 0 & \frac{1}{3} & \frac{2}{3} & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -\frac{1}{6} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{6} \\ \frac{1}{2} & -1 & \frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 6 & \frac{2}{3} & \frac{1}{6} & 0 \end{pmatrix} \begin{pmatrix} Q_{i-1} \\ Q_i \\ Q_{i+1} \\ Q_{i+2} \end{pmatrix}$

$$= \begin{pmatrix} \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & 0 \\ 0 & \frac{2}{3} & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \end{pmatrix} \begin{pmatrix} Q_{i-1} \\ Q_i \\ Q_{i+1} \\ Q_{i+2} \end{pmatrix}$$